CS 263: Counting and Sampling

Nima Anari

Stanford University

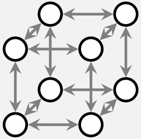
slides for

Comparison Arguments

Review

Example: hypercube

- Eigvals: k/n
- $\binom{n}{k}$ many



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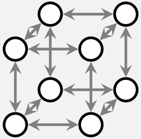

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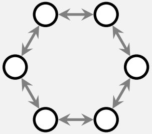

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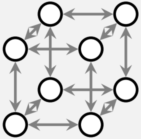

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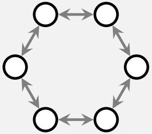

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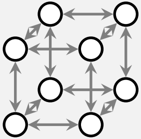

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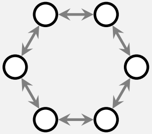

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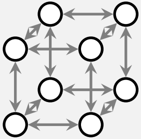

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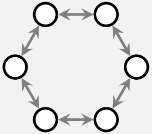

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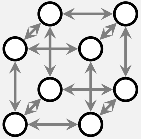

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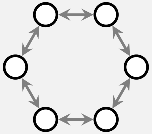

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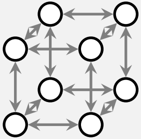

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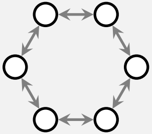

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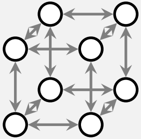

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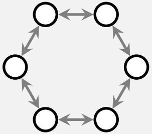

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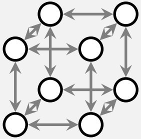

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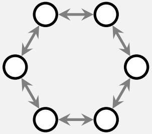

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- Comparison method

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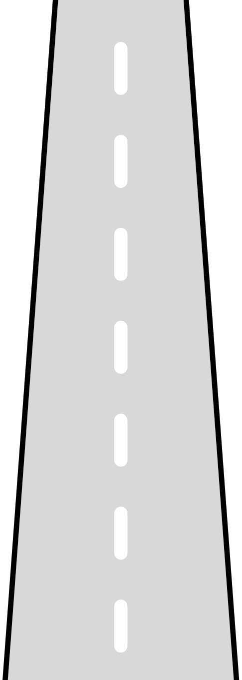
- Matchings



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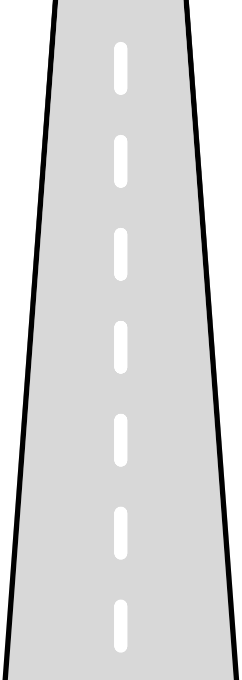

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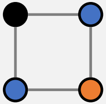

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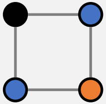

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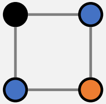

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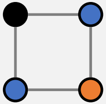

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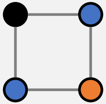

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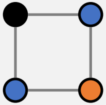

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- So as long as $Q(x,y) \geqslant c \cdot Q'(x,y)$:

$$\mathcal{E}_P(\phi'(f), f) \geqslant c \cdot \mathcal{E}_{P'}(\phi'(f), f)$$

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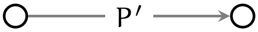

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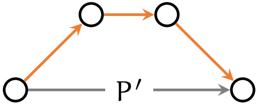

- Main application: when P' is the ideal chain, i.e.,

$$P' = \mathbb{1}\mu$$

col vec row vec

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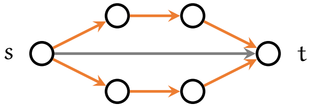

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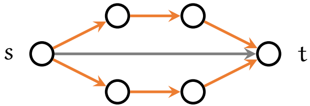

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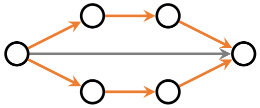

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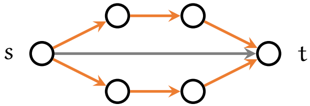

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Example: trivial routing

When $\pi = Q'$, length is 1 and congestion is

$$\max\left\{ \frac{Q'(x,y)}{Q(x,y)} \right\} = \max\left\{ \frac{P'(x,y)}{P(x,y)} \right\}$$

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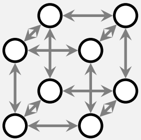

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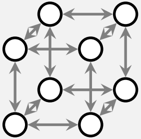

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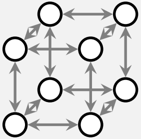

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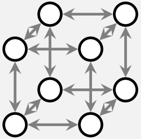

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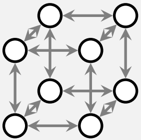

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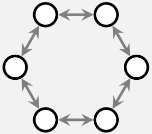

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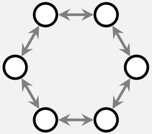

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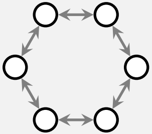

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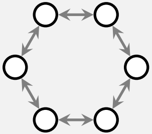

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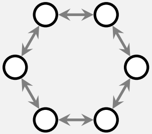

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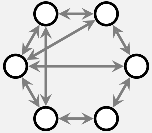

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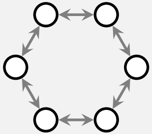

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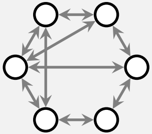

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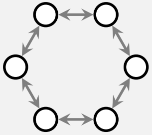

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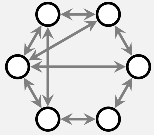

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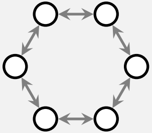

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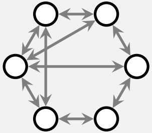

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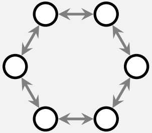

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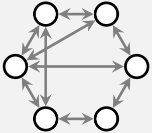

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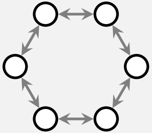

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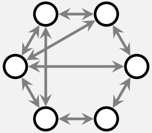

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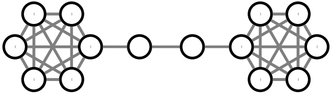

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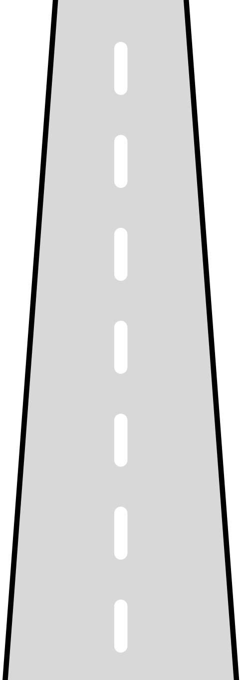
- Direct comparison
- Routing
- Comparison method

Applications

- Canonical paths
- Matchings



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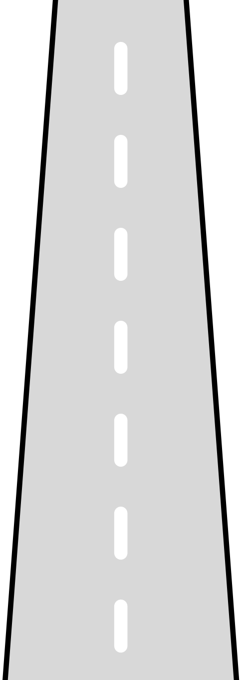

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to $\Omega \times [M]$: $\longleftarrow$ junk/side info

$$(s,t) \mapsto (r, \text{junk}).$$

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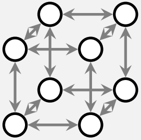

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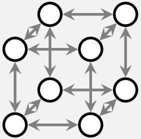

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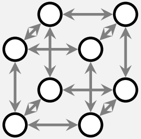

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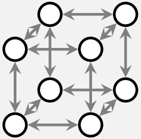

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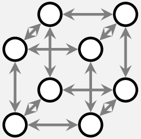

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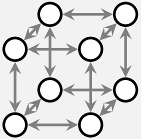

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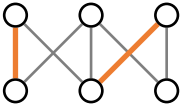
Unweighted graph, count/sample matchings.



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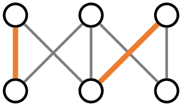

Markov chain (proposed by [Broder])

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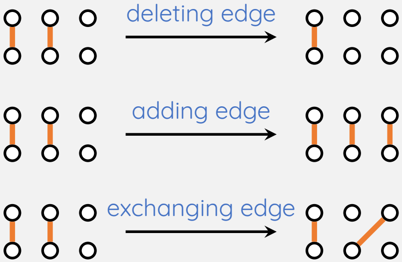

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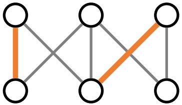

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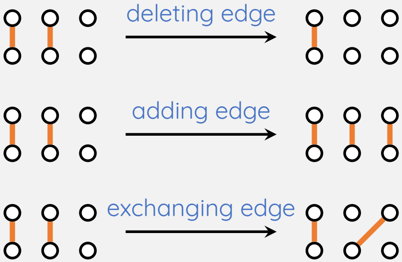

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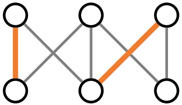

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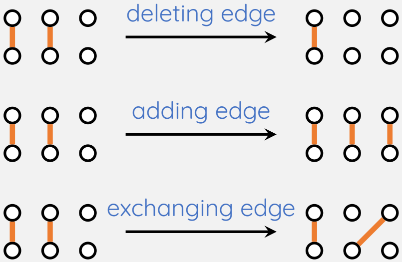

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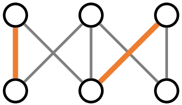

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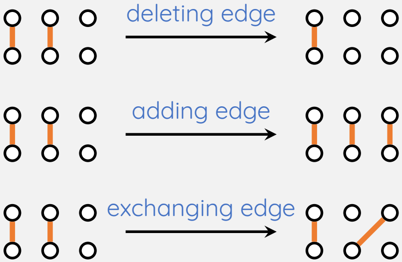

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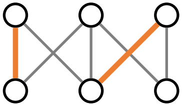

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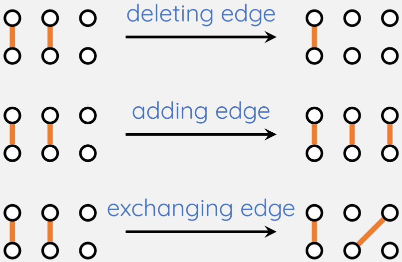

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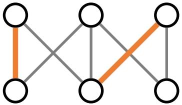

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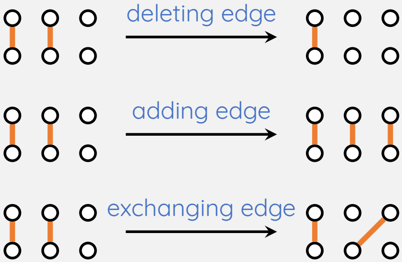

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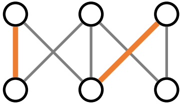

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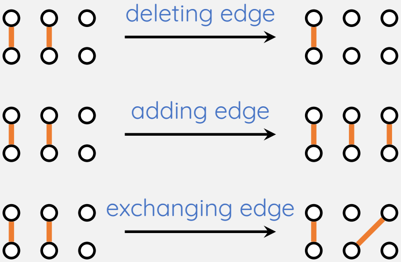

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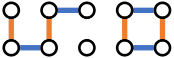

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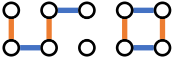

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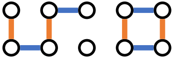

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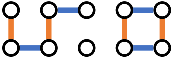

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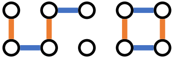

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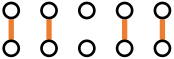

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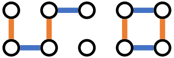

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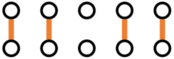
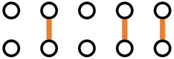
1  2 

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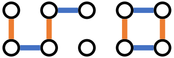

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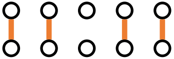

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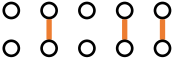

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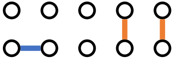

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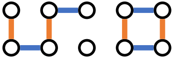

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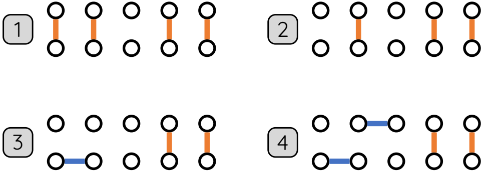

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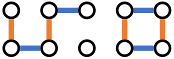

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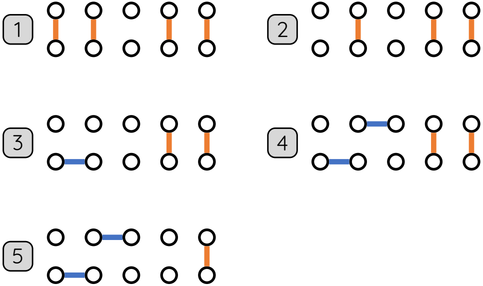

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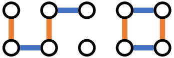
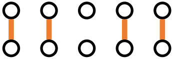


- This is a collection of alternating paths and cycles.
- We move from s to t one path/cycle at a time and unravel each path/cycle vertex-by-vertex.
- To make it deterministic, we fix an arbitrary order on all paths/cycles. For each cycle, we also fix an arbitrary start location.

- Example: let's unravel path, then cycle, and start cycle from top-left:

1


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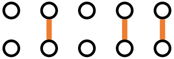

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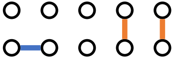

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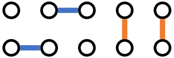

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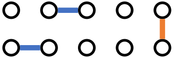
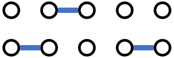


6


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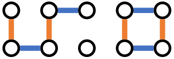

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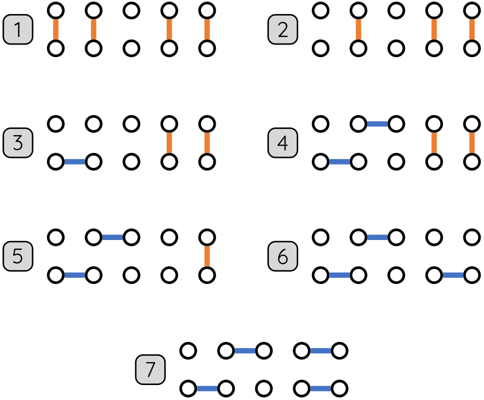

- For $x \to y$ transition, we can define encoding:

$$\text{enc}(s, t) = (s \oplus t \oplus x - \underbrace{\text{couple of edges}}_{\text{around current vertex}}, \underbrace{\text{couple of edges}}_{\text{junk/side info}})$$

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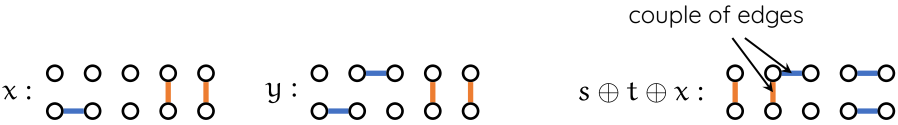

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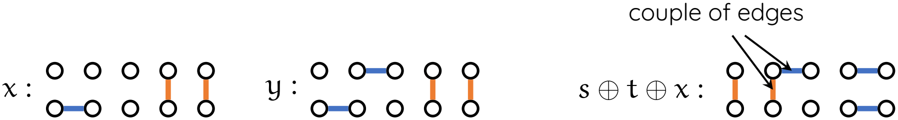

- Injective because we can recover $s \oplus t \oplus x$ from $\mathsf{enc}(s, t)$ and thus $s \oplus t$. So we can start unraveling x backward to get s and forward to get t.

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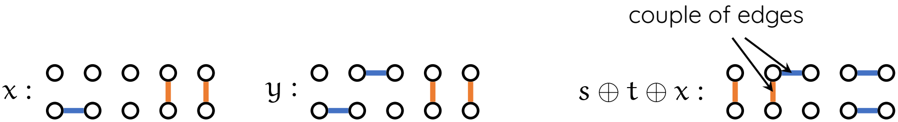

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- Thus the chain mixes in $\mathrm{poly}(n)$ time. 🙂