CS 263: Counting and Sampling

Nima Anari

Stanford University

slides for

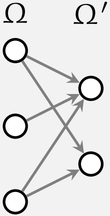
Dobrushin's Influence Matrix

Review

Markov kernel

$$N \in \mathbb{R}_{\geqslant 0}^{\Omega \times \Omega'}$$

$$\sum_y N(x, y) = 1$$



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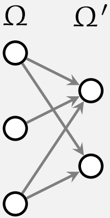

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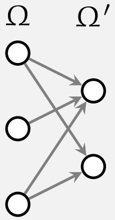

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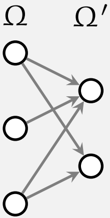

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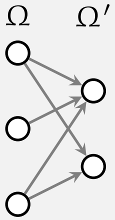

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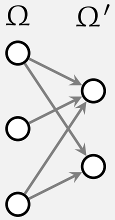

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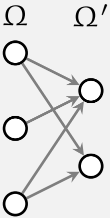

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- Mixing when $q \geqslant 4\Delta + 1$ 🙂

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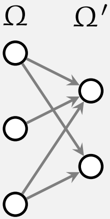

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Path coupling lemma

Suppose for all adjacent $X_0 \sim X_0'$ we can couple X_1, X_1' s.t.

$$\mathbb{E}[d(X_1,X_1')] \leqslant (1-c)d(X_0,X_0').$$

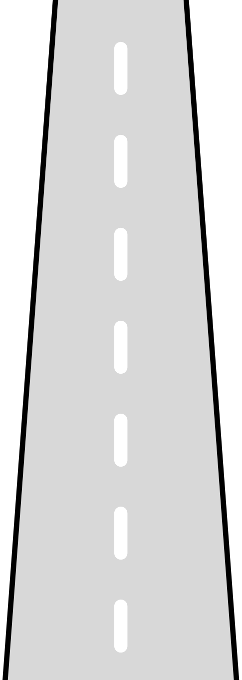
Then $\mathcal{W}(\nu P, \nu' P) \leqslant (1-c)\,\mathcal{W}(\nu,\nu')$.

Mixing via Transport

- Path coupling
- Dobrushin's condition
- Hardcore model
- Ising model
- Dobrushin++

Intro to Functional Analysis

- Divergences



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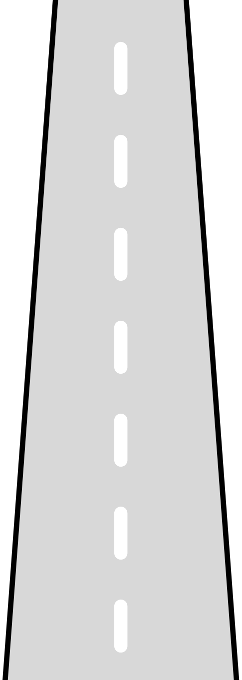

Path coupling for colorings [Jerrum]

Take the Metropolis chain for colorings:

- Pick u.r. vertex v
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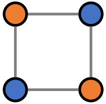
- Color v with c if valid



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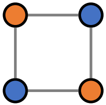

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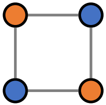

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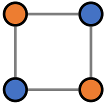

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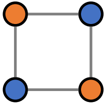

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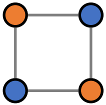

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- Let $a = X_0(w), b = X_0'(w)$. If $v \sim w$, use this coupling for color:

$$c: \quad 1 \quad 2 \cdots a \cdots b \cdots q$$

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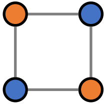

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which simplifies to

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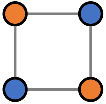

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$$t_{\mathrm{mix}}(\epsilon) = O\left(\frac{q}{q - 2\Delta} \cdot n \log(n/\epsilon)\right)$$

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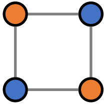

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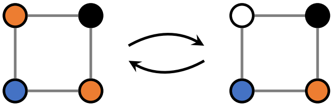
- What about Glauber?

Dobrushin influence matrix

Glauber for μ on $\Omega_1 \times \cdots \times \Omega_n$:

- Pick u.r. coord $i \in [n]$
- Replace coord i's value w.p. $\propto \mu(\text{result})$



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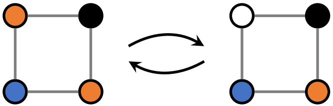

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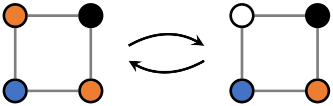

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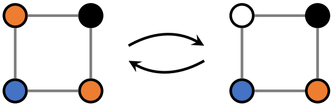

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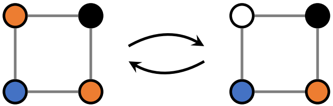

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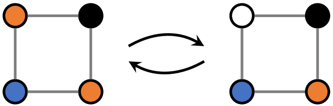

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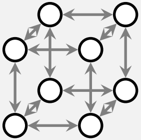
Example: hypercube

- $\Omega = \{0, 1\}^n$
- $\mu(x) = \text{uniform}$



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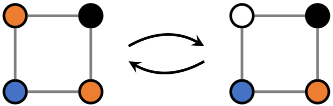

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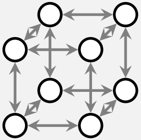
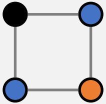


Example: coloring

- $\Omega = [q]^n$
- $\mu = \text{proper coloring}$



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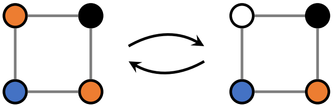

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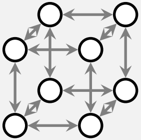

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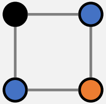

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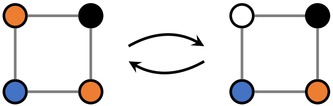


- We pick $\omega \in \Omega_i$ from $\mathsf{dist}(X_i \mid X_{-i})$.
- Influence: take worst case X, X' that differ in coord j:

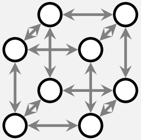
$$d_{TV}\big(\mathsf{dist}(X_i \mid X_{-i}), \mathsf{dist}(X'_i \mid X'_{-i})\big)$$

- Call maximum value $\mathcal{I}[j \to i]$.

- Dobrushin influence matrix: matrix with entries $\mathcal{I}[j \to i]$.

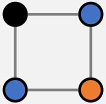
Example: hypercube

- $\Omega = \{0, 1\}^n$
- $\mu(x) = \text{uniform}$
- $\mathcal{I}[j \to i] = 0$



Example: coloring

- $\Omega = [q]^n$
- $\mu = \text{proper coloring}$
- $\mathcal{I}[j \to i] = 0$ when $j \not\sim i$
- $\mathcal{I}[j \to i] \leqslant 1/(q - \Delta)$



Dobrushin's condition

If columns of $\mathcal{I}$ sum to $\leqslant 1-\delta$, then

$$\mathcal{W}(\nu P, \nu' P) \leqslant (1-\delta/n)\, \mathcal{W}(\nu, \nu')$$

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If columns of $\mathcal{J}$ sum to $\leqslant 1-\delta$, then

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$$t_{\mathsf{mix}}(\epsilon) = O\left(\tfrac{1}{\delta} n \log(n/\epsilon)\right)$$

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$$\frac{\Delta}{q-\Delta} < 1 \;\leftrightarrow\; q \geqslant 2\Delta + 1$$

same cond as Metropolis

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the one defining d_{TV}

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  - 2 (different replacements)

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- Possibilities for $d(X_1, X_1')$:
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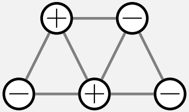
  - 2 (different replacements)
- We get $\mathbb{E}[d(X_1, X_1')] \leqslant \frac{1}{n} \cdot 0 + \frac{1}{n} \sum_{i \neq j} (1 + \mathcal{J}[j \to i]) \leqslant 1 - \delta/n$

- Useful for spin systems.

▷ Useful for spin systems.

Spin systems



graph $G = (V, E)$

$$\Omega = \Omega_1 \times \cdots \times \Omega_n$$

local interaction

$$\mu(x) = \prod_v \phi_v(x_v) \cdot \prod_{u \sim v} \phi_{uv}(x_u, x_v)$$

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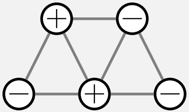
Spin systems



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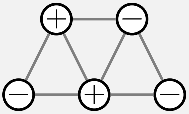

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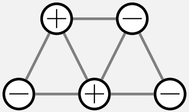

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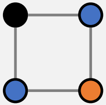

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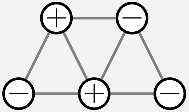

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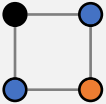
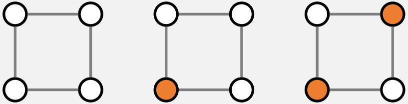


Example: hardcore



$\mu \propto 1$ $\quad$ $\mu \propto \lambda$ $\quad$ $\mu \propto \lambda^2$

$\Omega_i = \{0, 1\}$ $\quad$ $\phi_v = \lambda^{x_v}$ $\quad$ $\phi_{uv} = 1 - x_u x_v$

fugacity

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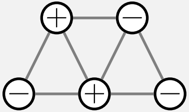
Spin systems



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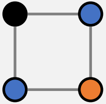

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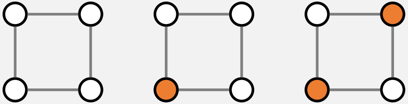

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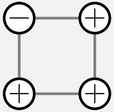
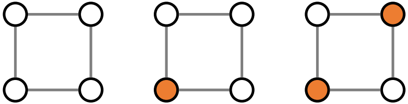
Example: Ising

- $\Omega_i = \{\pm 1\}$
- $\phi_v = \exp(\underbrace{h_v}_{\text{external field}} x_v)$
- $\phi_{uv} = \exp(\underbrace{\beta_{uv}}_{\text{ferro/anti-ferromagnetic interaction}} x_u x_v)$



Hardcore model



$\mu(\text{ind set } S) \propto \lambda^{|S|}$

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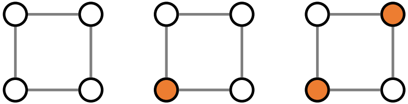

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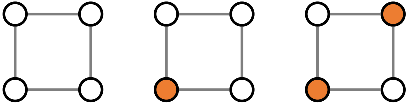

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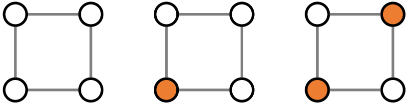

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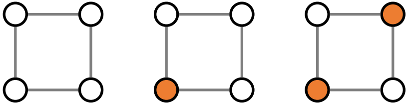

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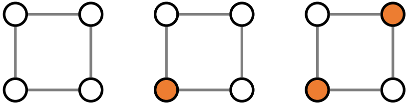

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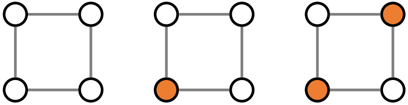

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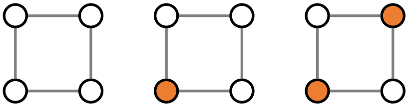

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  for a specific critical threshold

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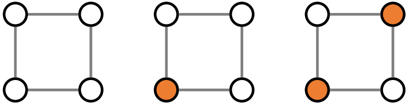

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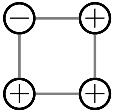
- On the opposite side, [Sly'10] showed it is NP-hard to sample when

  $$\lambda \geqslant (1+\delta)\lambda_c(\Delta)$$

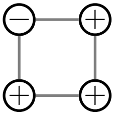
Ising model



$$\mu(x) \propto \exp\left(\tfrac{1}{2} \sum_{u,v} \beta_{uv} x_u x_v + \sum_v h_v x_v\right)$$

symmetric matrix

Ising model



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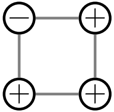

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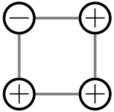

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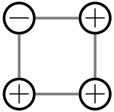

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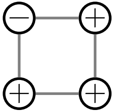

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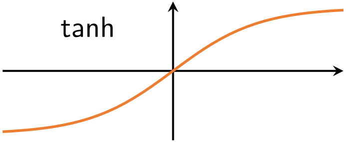

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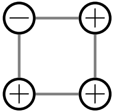

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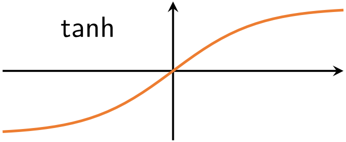

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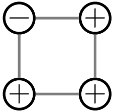

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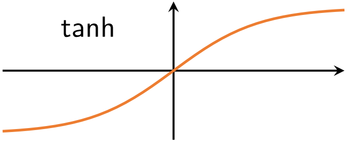

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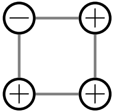

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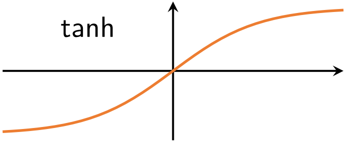

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- Can be asymptotically tight for certain β (Curie-Weiss).

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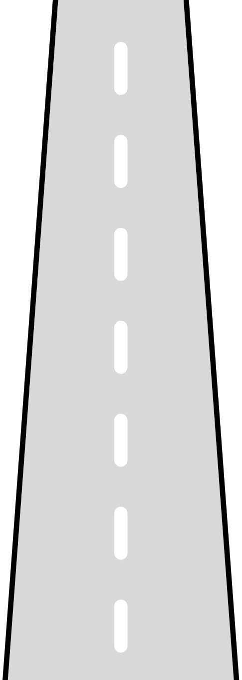
- We can apply Dobrushin++ as long as $\lambda_{\max}(\mathcal{I}) < 1$

Mixing via Transport

- Path coupling
- Dobrushin's condition
- Hardcore model
- Ising model
- Dobrushin++

Intro to Functional Analysis

- Divergences



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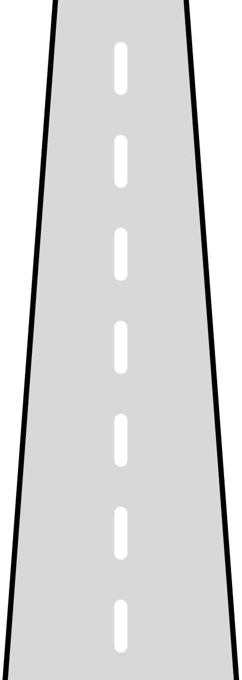

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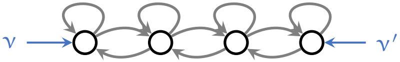

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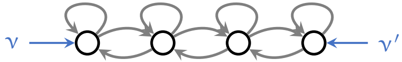

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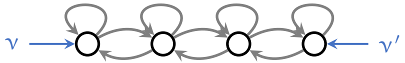

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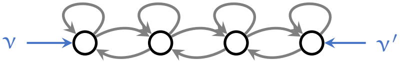

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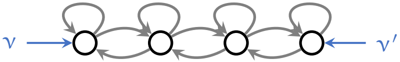

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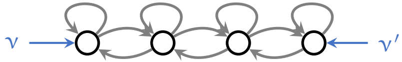

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usually f in the literature

ϕ-divergence

For measure ν and dist μ define

$$\mathcal{D}_\phi(\nu \parallel \mu) = \mathrm{Ent}_\mu^\phi\left[\frac{\nu}{\mu}\right]$$

Proxy for d_{TV}

Contraction: $\mathcal{D}_\phi(\nu P \parallel \mu) \leqslant (1-\delta)\, \mathcal{D}_\phi(\nu \parallel \mu)$ for stationary μ.

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Variance

$$\phi(x) := x^2$$

- $\mathrm{Ent}_\mu^\phi[f] = \mathrm{Var}_\mu[f]$
- $\mathcal{D}_\phi(\nu \parallel \mu) = \chi^2(\nu \parallel \mu)$
- It is a proxy by Cauchy-Schwarz:

$$d_{TV}(\nu, \mu) \leqslant O\left(\sqrt{\chi^2(\nu \parallel \mu)}\right)$$

- Contraction related to eigs of P.

called Poincaré inequality

Entropy

$$\phi(x) := x \log x$$

- $\mathrm{Ent}_\mu^\phi[f] = \mathrm{Ent}_\mu[f]$
- $\mathcal{D}_\phi(\nu \parallel \mu) = \mathcal{D}_{KL}(\nu \parallel \mu)$
- It is a proxy by Pinsker:

$$d_{TV}(\nu, \mu) \leqslant O\left(\sqrt{\mathcal{D}_{KL}(\nu \parallel \mu)}\right)$$

- Contraction: very hard!

called modified log-Sobolev inequality