CS 263: Counting and Sampling

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Stanford University

slides for

Det Counting; Markov Chains

Review

- DNF counting:

$$\underbrace{|A_1 \sqcup \cdots \sqcup A_m|}_{\text{easy to compute}} \cdot \underbrace{\frac{|A_1 \cup \cdots \cup A_m|}{|A_1 \sqcup \cdots \sqcup A_m|}}_{\text{probability}}$$

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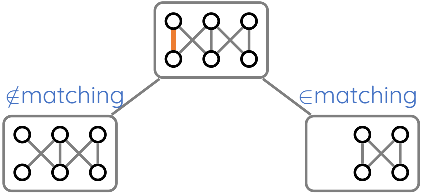

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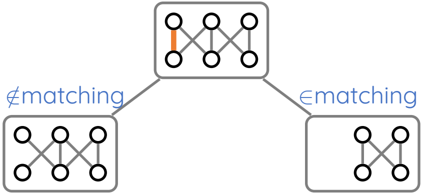

$$\begin{tikzcd}
 & \text{(FPRAS)} \\
\text{Exact Counting} \arrow[r] \arrow[d] & \text{Approx Counting} \arrow[dl, "\text{if FPTAS}"'] \arrow[d, leftrightarrow] \\
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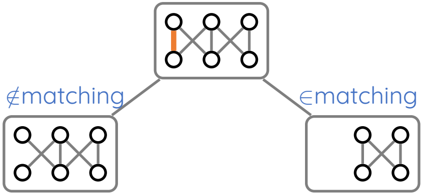

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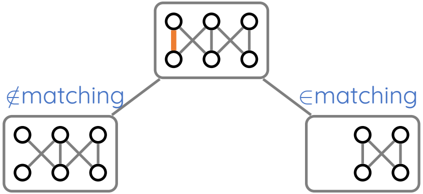


(FPRAS)

Exact Counting ⟶ Approx Counting

Exact Sampling ⟶ Approx Sampling

if FPTAS

(FPAUS)

- Coupling: dist with marginals μ, ν.
- Matrix-tree theorem [Kirchhoff]:

$$\#\text{spanning trees} = \det(\text{matrix})$$

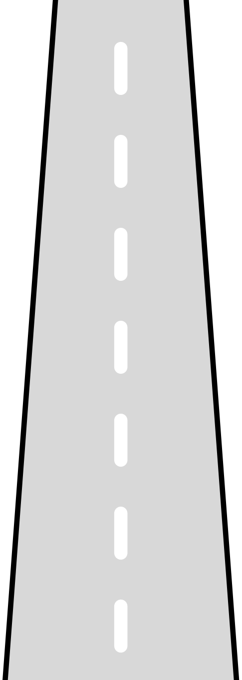
Laplacian, drop one row+col

Counting via Determinants

- Spanning trees
- Bipartite planar perfect matchings

Intro to Markov Chains

- Stationary distribution
- Fundamental theorem
- Mixing time



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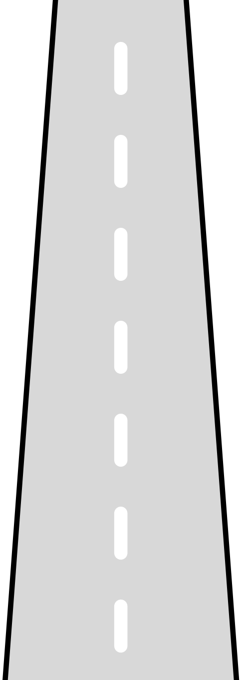

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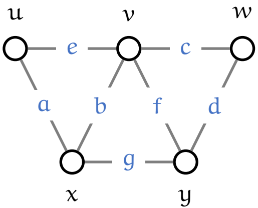

$$\begin{array}{c} \\ u \\ v \\ w \\ x \\ y \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} & \mathrm{g} \end{array} \\
\begin{bmatrix}
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vertex-edge adj matrix A

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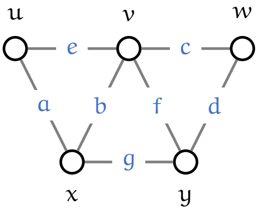

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If we take $AA^\intercal$, we get the Laplacian:

$$(AA^\intercal)_{ij} = \begin{cases} -\mathbb{1}[i \sim j] & \text{if } i \neq j, \\ \deg(i) & \text{if } i = j. \end{cases}$$

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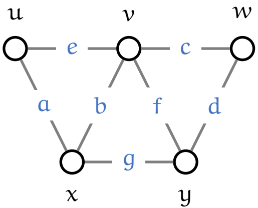

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det of $(n-1) \times (n-1)$ principal submatrix of Laplacian is #spanning trees.

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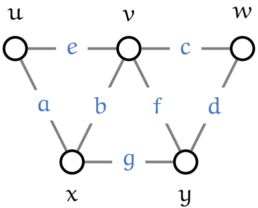

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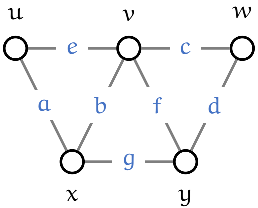

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- Counting $\implies$ sampling.

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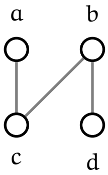

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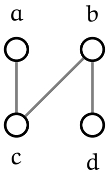

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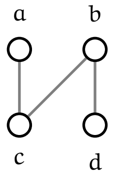

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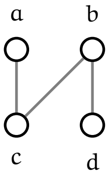

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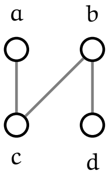

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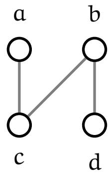

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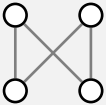
Example: $K_{2,2}$



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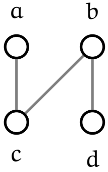

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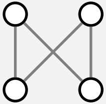

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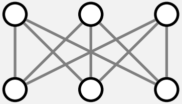
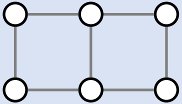
Non-example: $K_{3,3}$



Impossible! Exercise: show this.

Theorem [Fisher-Kasteleyn-Temperley]

If graph is planar, [Pólya]'s scheme can be implemented. E.g., 2D lattice:



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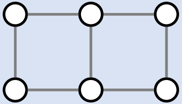

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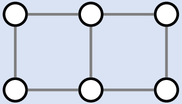

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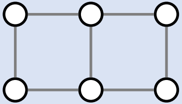

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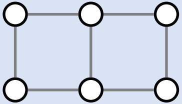

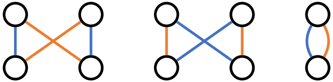

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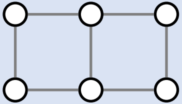

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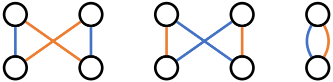

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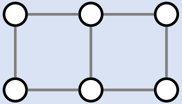

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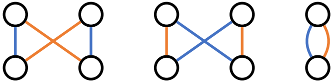

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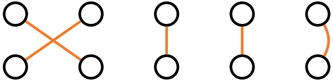

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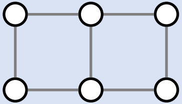

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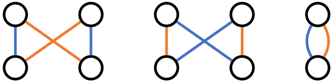

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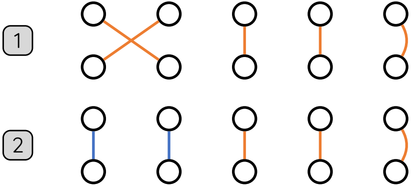

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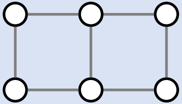

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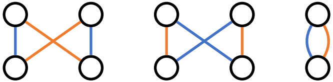

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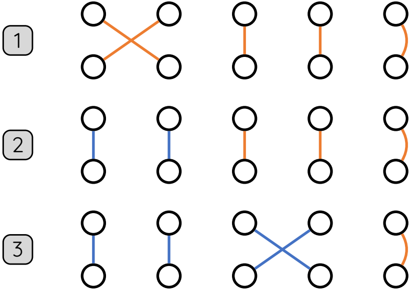

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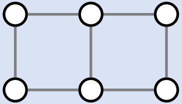

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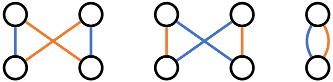

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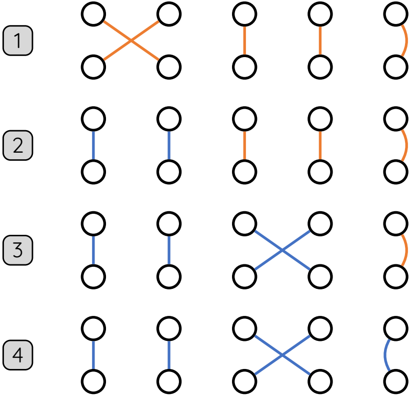

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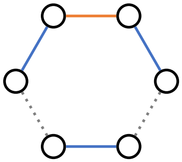

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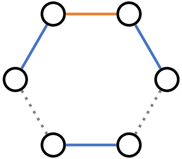

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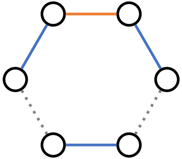

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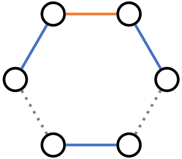

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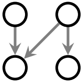
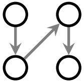
 $\begin{bmatrix} + & 0 \\ + & + \end{bmatrix}$  $\begin{bmatrix} + & 0 \\ - & + \end{bmatrix}$

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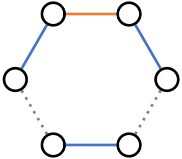

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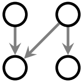

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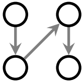

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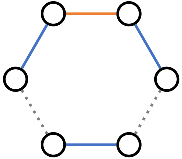

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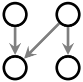
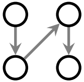
 $\begin{bmatrix} + & 0 \\ + & + \end{bmatrix}$  $\begin{bmatrix} + & 0 \\ - & + \end{bmatrix}$

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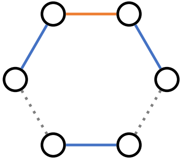

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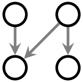
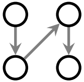
 $\begin{bmatrix} + & 0 \\ + & + \end{bmatrix}$  $\begin{bmatrix} + & 0 \\ - & + \end{bmatrix}$

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- This means: 🙂

$$\prod_{e\in \mathsf{cycle}} A_e = (-1)^{\mathsf{len}/2+1}$$

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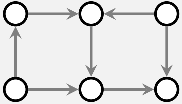

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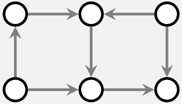

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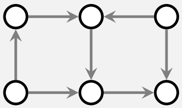

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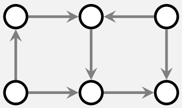

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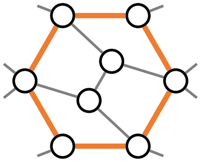

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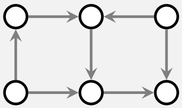

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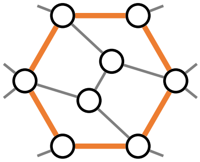

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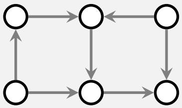

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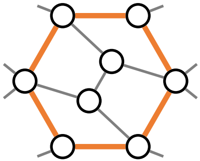

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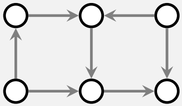

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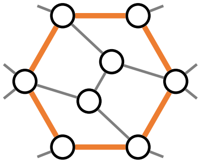

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- Because of niceness, there are even many interior vertices. ☺

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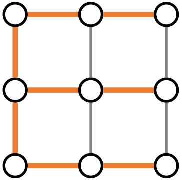

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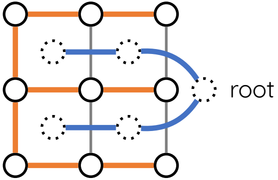

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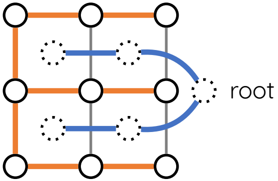

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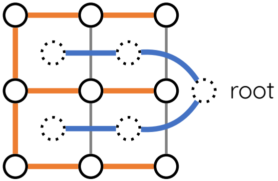

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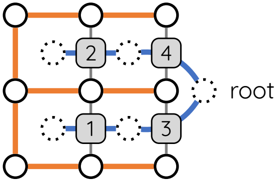

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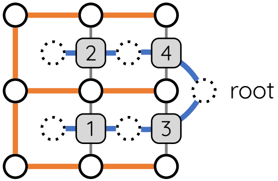

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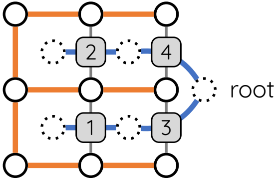

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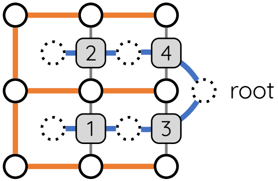

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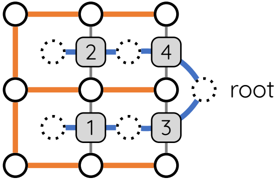

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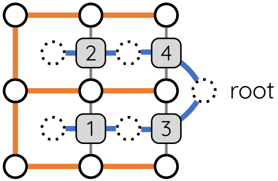

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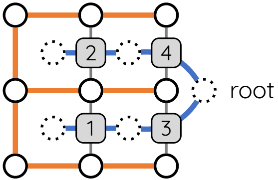
- How to make faces happy?
- Choose spanning tree
- Comes with dual spanning tree



- Orient spanning tree arbitrarily.
- Peel leaves of dual spanning tree one-by-one. Each time, orient the single remaining edge of peeled face uniquely.

Summary: counting via dets

- Spanning trees:
  - Undirected: [Kirchhoff]'s matrix-tree theorem.
  - Directed: exercise!
- Planar perfect matchings:
  - Bipartite: [Fisher-Kasteleyn-Temperley]'s Pfaffian orientation.
  - Non-bipartite: exercise!
- Holographic reductions [Valiant]
- Eulerian tours: exercise!
- Determinantal point processes

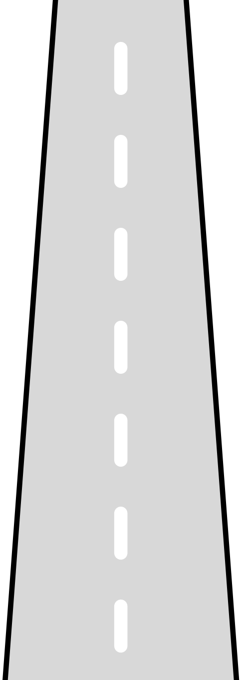
will see later

Counting via Determinants

- Spanning trees
- Bipartite planar perfect matchings

Intro to Markov Chains

- Stationary distribution
- Fundamental theorem
- Mixing time



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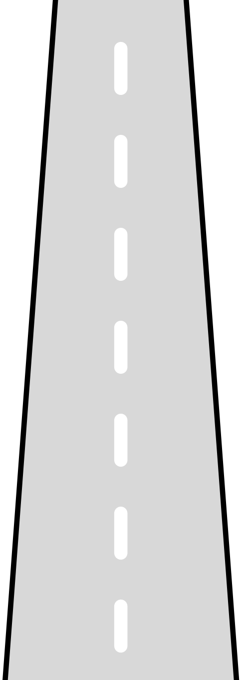

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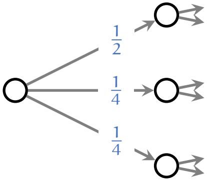

space Ω

Transition matrix: $P \in \mathbb{R}_{\geqslant 0}^{\Omega \times \Omega}$

large and implicit

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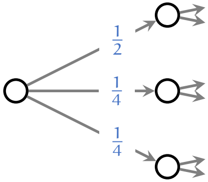

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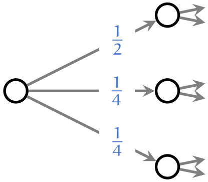

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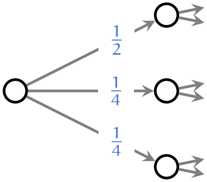

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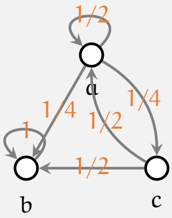

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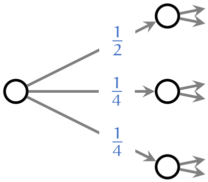

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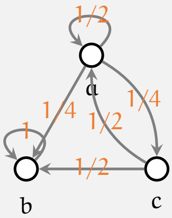

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- Given (random) start X_0, we get Markovian process:

$$X_0 \to X_1 \to X_2 \to \dots$$

transition via P　　transition via P

Fundamental theorem

Under “mild conditions”:

$$\mathrm{dist}(X_t) \to \mu$$

where μ is the stationary dist.

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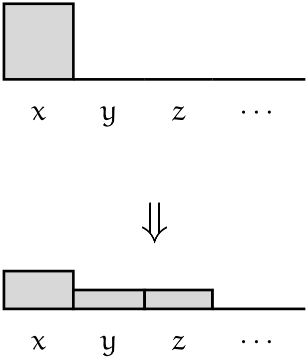

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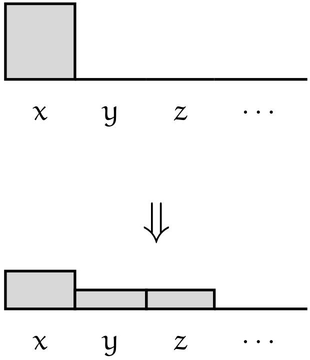

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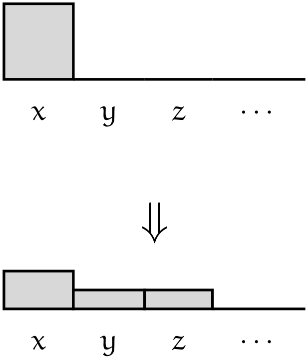

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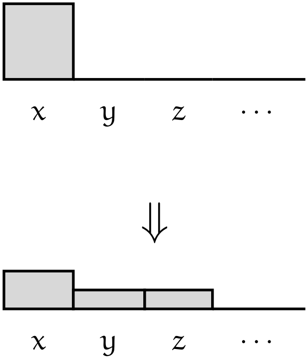

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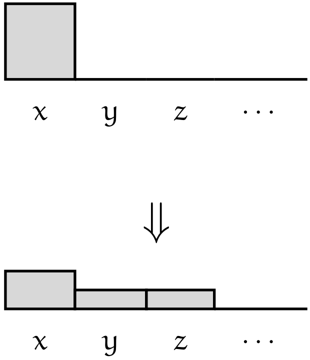

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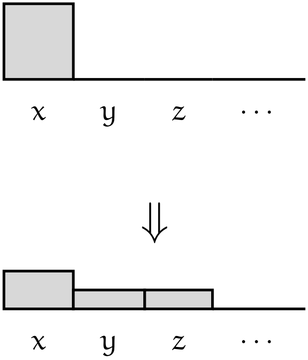

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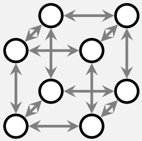
- Ideally, we want to stop at small t and have small d_{TV} to μ.

Example: hypercube

- $\Omega = \{0,1\}^n$
- Pick u.r. $i \in [n]$
- Replace coord i with $\mathrm{Ber}(\frac{1}{2})$



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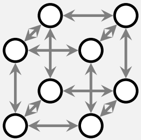

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- $\Omega =$ valid colorings
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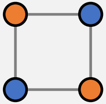

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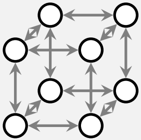

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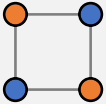

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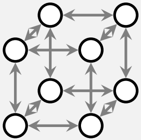

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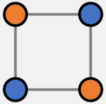

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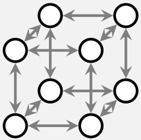

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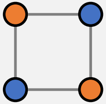

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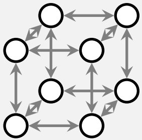

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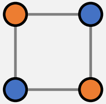

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$$\lim_{t\to\infty} \nu P^t = \mu.$$

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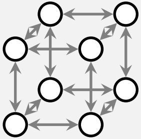

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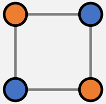

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- Note: this convergence can be very slow.

Much more useful for us:

Mixing time

For Markov chain P with stationary μ, we set

$$t_{\text{mix}}(P, \epsilon, \nu) = \min\left\{t \;\middle|\; d_{\text{TV}}(\mu, \nu P^t) \leqslant \epsilon\right\}$$

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- We usually want $t_{\text{mix}}(P) = \text{poly}\log(|\Omega|)$ for efficient algs.