CS 263: Counting and Sampling

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slides for

Sampling vs. Counting

Review

Density μ on space Ω

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- Sampling: $\mathbb{P}[\text{output}] \propto \mu(\text{output})$

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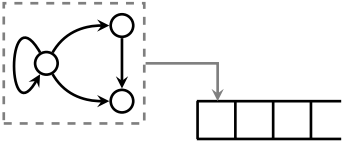

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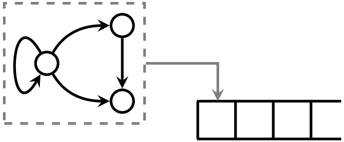

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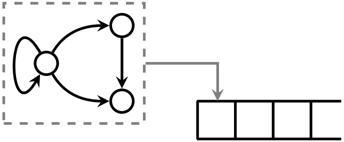

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Approx counting

$(1+\epsilon)$-approx in poly$(n, 1/\epsilon)$

FPRAS/FPTAS

randomized deterministic

Approx sampling

δ-approx in d_{TV} in poly$(n, \log(1/\delta))$

FPAUS

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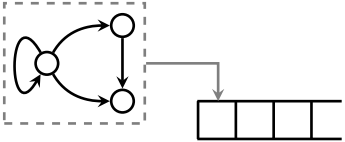

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FPRAS/FPTAS	FPAUS

randomized (FPRAS) deterministic (FPTAS)

Self-reducibles [Jerrum-Valiant-Vazirani]:

Exact Counting → Approx Counting

Exact Counting ↓ Exact Sampling

Approx Counting → Exact Sampling (if FPTAS)

Approx Counting ↕ Approx Sampling

Exact Sampling → Approx Sampling

DNF Counting

- Rejection sampling
- Monte Carlo estimation

Counting vs. Sampling

- Self-reducibility
- Reductions
- Total variation and coupling

Counting via Determinants ← if time

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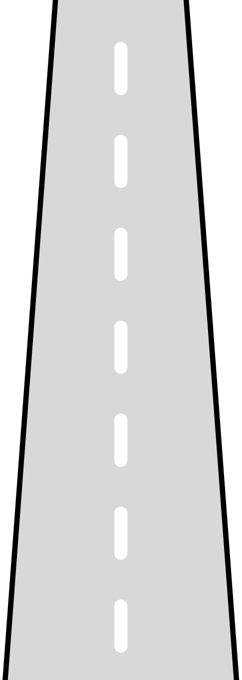

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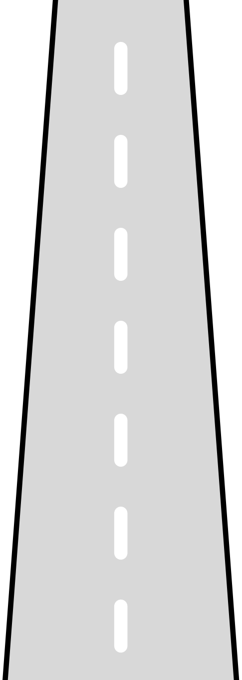
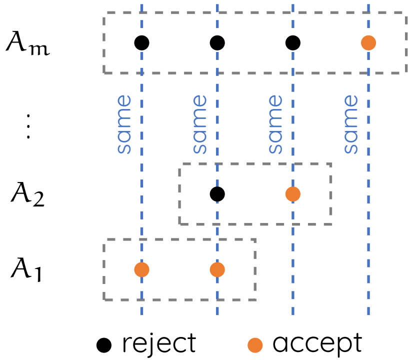


DNF sampling [Karp-Luby]

$$\phi = C_1 \vee C_2 \vee \cdots \vee C_m$$

- $A_i = \{$sat assignments of $C_i\}$
- Sample u.r. $\in A_1 \sqcup \cdots \sqcup A_m$



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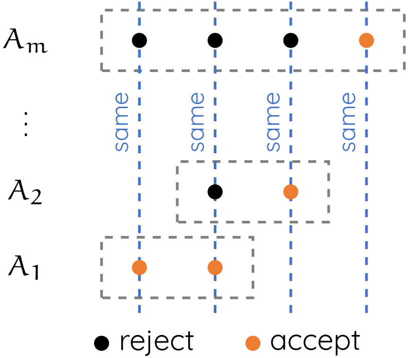

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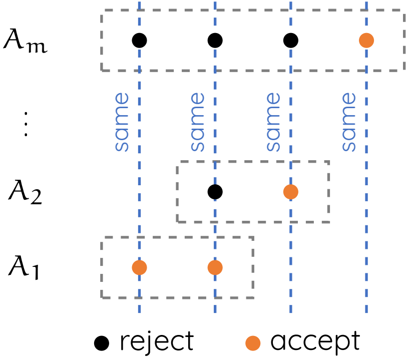

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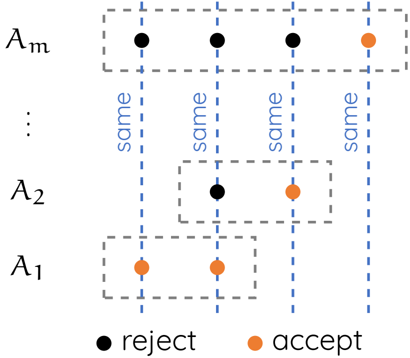

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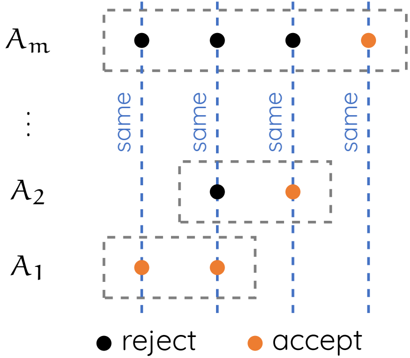

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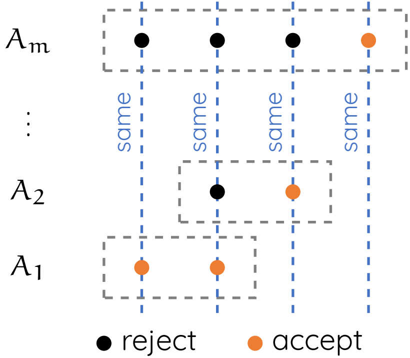

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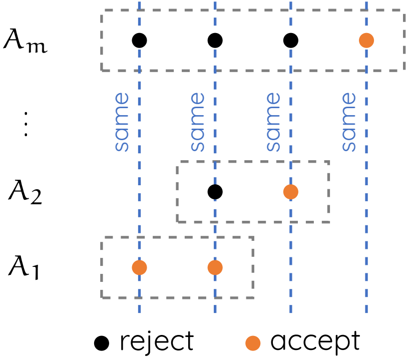

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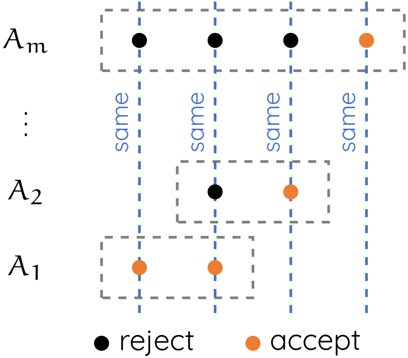

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- Idea: Write $|A_1 \cup \cdots \cup A_m|$ as

$$\underbrace{|A_1 \sqcup \cdots \sqcup A_m|}_{\text{easy to compute}} \cdot \underbrace{\frac{|A_1 \cup \cdots \cup A_m|}{|A_1 \sqcup \cdots \sqcup A_m|}}_{\text{accept prob}}$$

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- Approximate accept prob p

Monte Carlo estimation

for $i = 1, \ldots, t$ **do**
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  and $X_i \leftarrow \mathbb{1}[\text{accept}]$

return $X = \frac{X_1 + \cdots + X_t}{t}$

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which is $\leqslant 9/tp\epsilon^2$.

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Lemma

To mult. estimate p from $\mathrm{Ber}(p)$ samples, $O(1/p\epsilon^2)$ many enough.

Open problem: Is there an FPTAS for DNF counting?

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[Gopalan-Meka-Reingold'12]

$\pm\epsilon 2^n$ approximation in time

$$n^{\widetilde{O}(\log \log n)}$$

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Counting vs. Sampling

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Counting via Determinants ← if time

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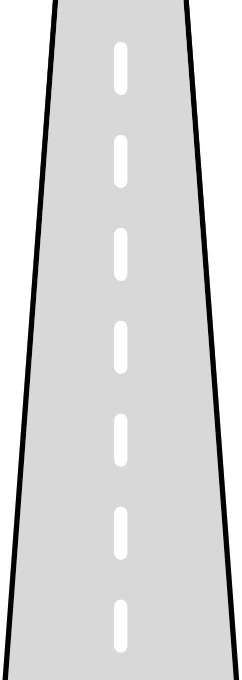

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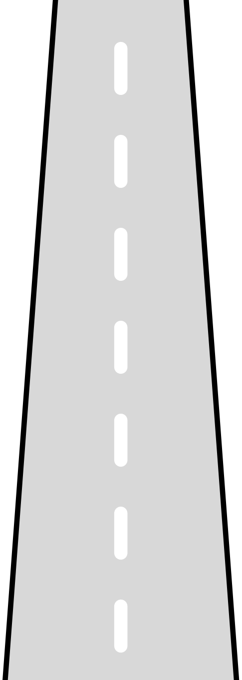
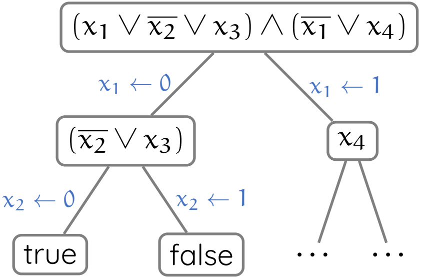


Self-reducible problems

advanced: measure-decomposed

Solutions of instance I partitioned. Each part $\equiv$ smaller instance I'.



Key: branching factor, depth $\leqslant$ poly

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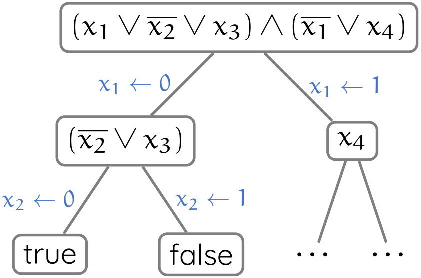

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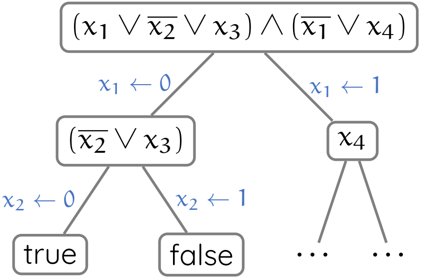

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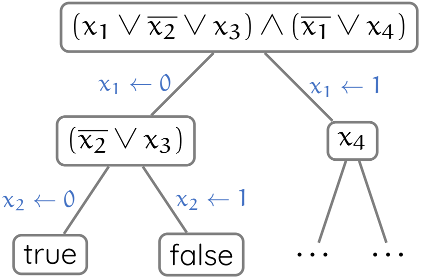

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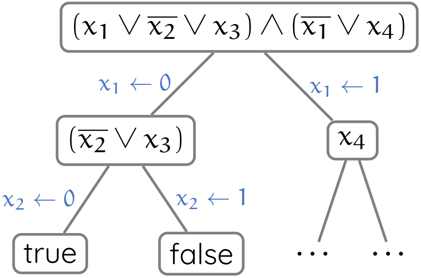

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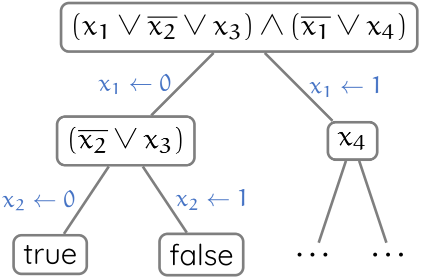

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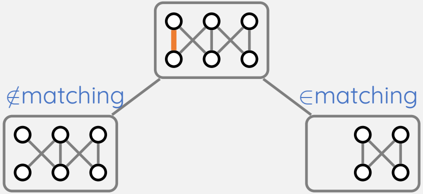
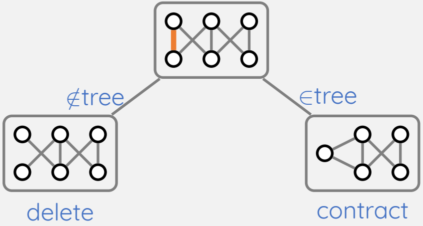
Example: perfect matchings



Example: spanning trees



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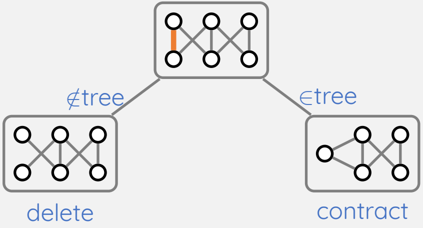

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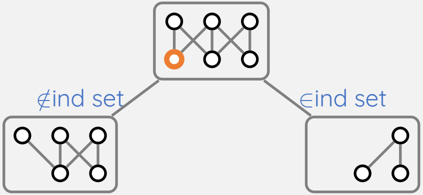

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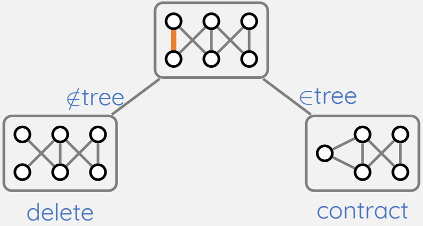

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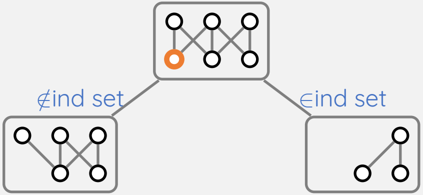
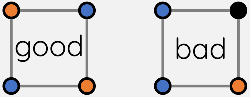


Non-example: colorings

Instance: graph $G = (V, E)$ and $q > 0$

Solutions: $x \in [q]^V$ with $x_u \neq x_v$ for adjacent u, v



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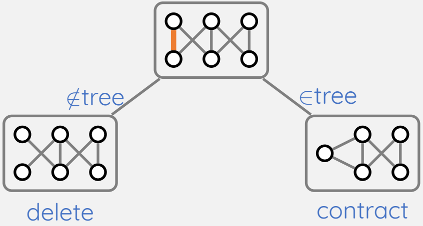

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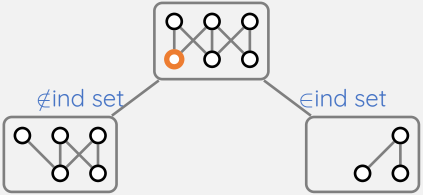

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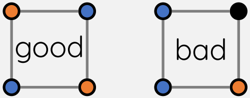
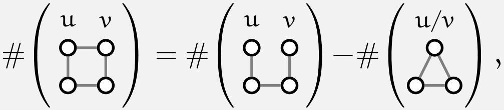


Note that



but this is not self-reducibility.

Theorem [Jerrum-Valiant-Vazirani]

For "self-reducible" problems:

approx counting $\equiv$ approx sampling

$$
\begin{tikzcd}
 & (\text{FPRAS}) \\
\text{Exact Counting} \arrow[r] \arrow[d] & \text{Approx Counting} \arrow[dl, "\text{if FPTAS}"'] \arrow[d, leftrightarrow] \\
\text{Exact Sampling} \arrow[r] & \text{Approx Sampling} \\
 & (\text{FPAUS})
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arrows are poly-time reductions

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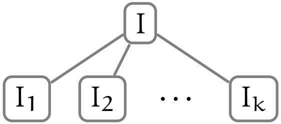
Exact Counting $\Longrightarrow$ Exact Sampling



(FPRAS)

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(FPAUS)

arrows are poly-time reductions

while I *not base case* **do**

  compute children $I_1, \dots, I_k$

  for $i = 1, \dots, k$ **do**

    $c_i \leftarrow \#(I_i)$

  choose i w.p. $\propto c_i$

  $I \leftarrow I_i$

output sample for I

$$\mathbb{P}[\text{sample}] = \frac{\#(I_i)}{\#(I)} \cdot \frac{\#(I_{ij})}{\#(I_i)} \cdots = \frac{1}{\#(I)}$$

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But we only want $d_{TV} \leqslant \delta$. ☺

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- Runtime: $\operatorname{poly}(n, \log(1/\delta))$ ☺

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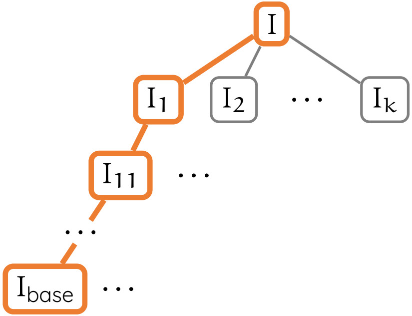

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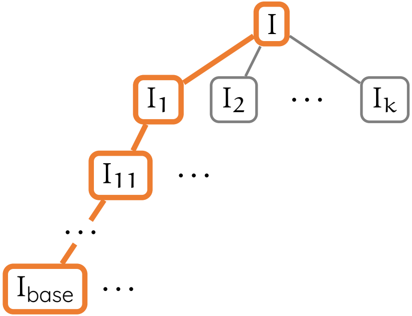

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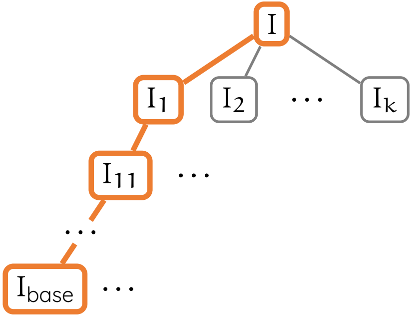

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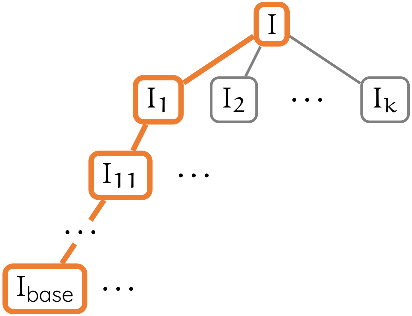

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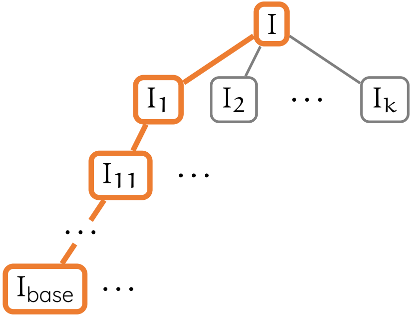

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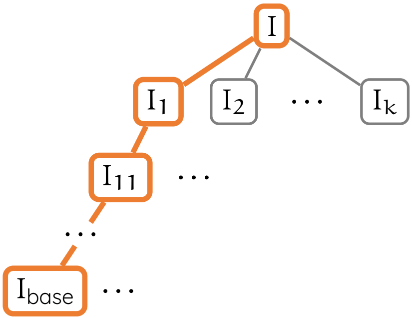

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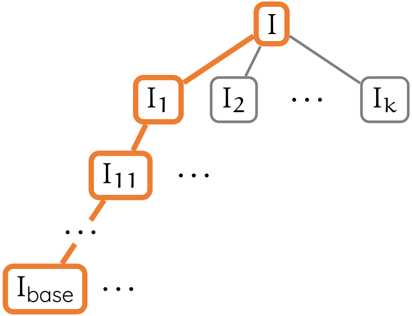

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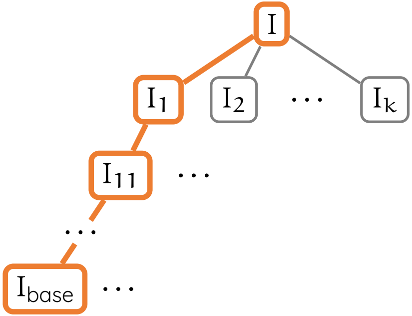

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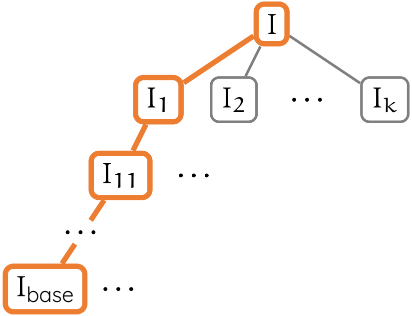

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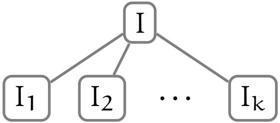
- Problem: if any ratio p is small, it takes $\geqslant 1/p$ time to estimate.



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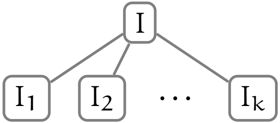

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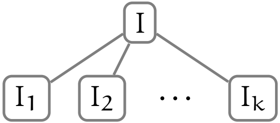

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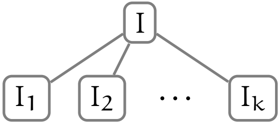

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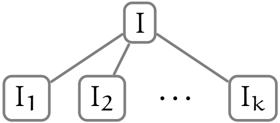

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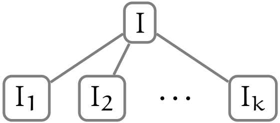

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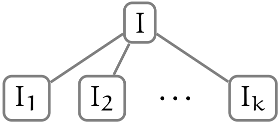

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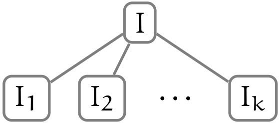

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Lemma

In a randomized poly-time algorithm, exact samplers can be replaced by FPAUS while guaranteeing the output changes no more than δ in d_{TV} at the cost of $\mathbf{poly}(n, \mathbf{log}(1/\delta))$ in runtime.

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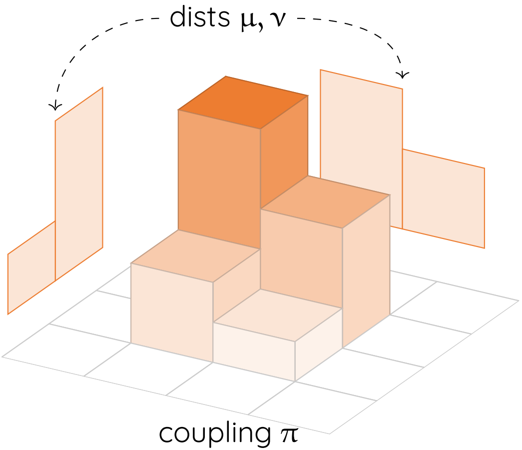
For dists μ, ν, a coupling is a joint dist π of (X, Y) where $X \sim \mu$ and $Y \sim \nu$.



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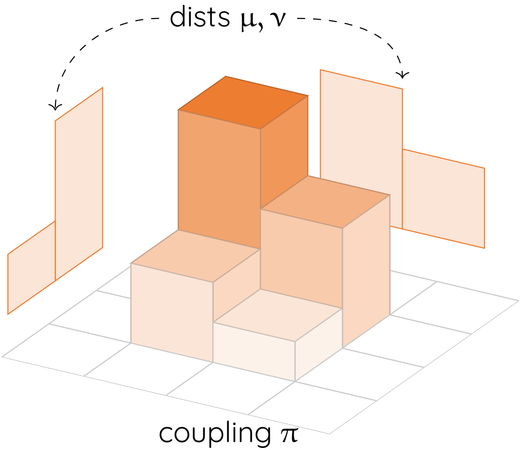

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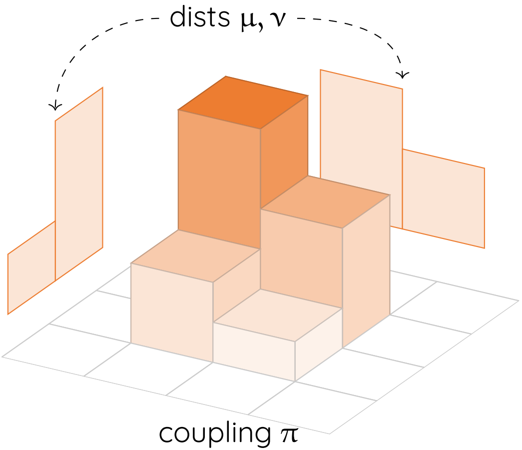

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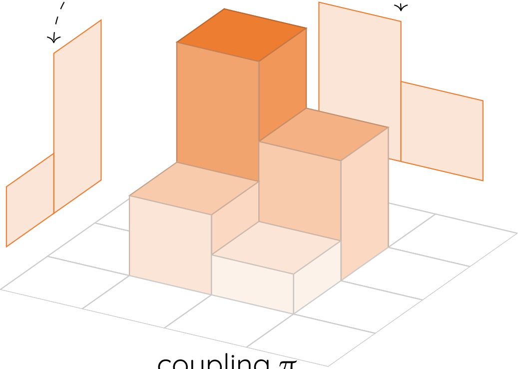

- Proof: exercise!
- Useful mindset: think of coupling as an alg to produce X, Y. Compose these algs together.

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DNF Counting

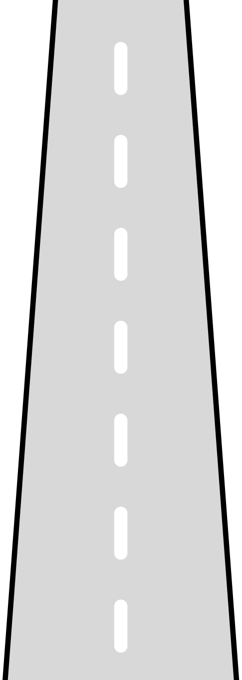
- Rejection sampling
- Monte Carlo estimation

Counting vs. Sampling

- Self-reducibility
- Reductions
- Total variation and coupling

Counting via Determinants ← if time

- Spanning trees



DNF Counting

- Rejection sampling
- Monte Carlo estimation

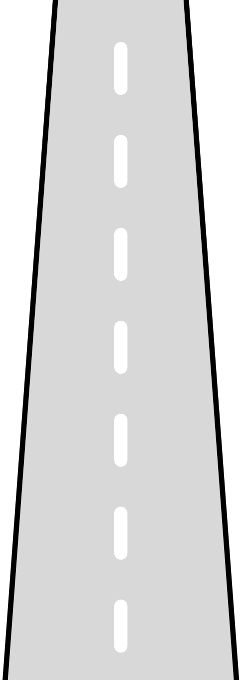
Counting vs. Sampling

- Self-reducibility
- Reductions
- Total variation and coupling

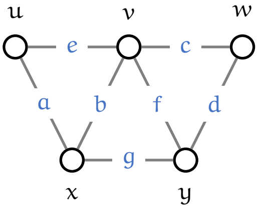
Counting via Determinants

if time

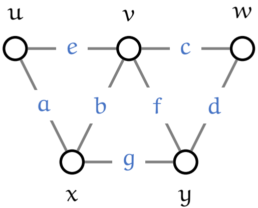
- Spanning trees



Counting spanning trees



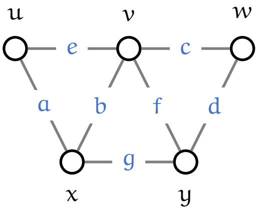
Counting spanning trees



$$
\begin{array}{c} \\ u \\ v \\ w \\ x \\ y \end{array}
\begin{array}{c}
\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{bmatrix}
+1 & 0 & 0 & 0 & +1 & 0 & 0 \\
0 & -1 & +1 & 0 & -1 & -1 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & +1 & +1
\end{bmatrix}
\end{array}
$$

vertex-edge adj matrix

Counting spanning trees



$$
\begin{array}{c|c}
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\begin{bmatrix}
+1 & 0 & 0 & 0 & +1 & 0 & 0 \\
0 & -1 & +1 & 0 & -1 & -1 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 & 0 & -1 \\
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\end{array}
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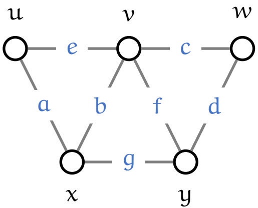
vertex-edge adj matrix

- Sum of rows $= 0$

Counting spanning trees

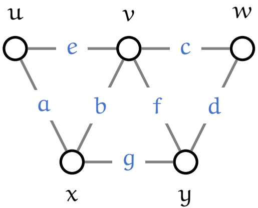


$$
\begin{array}{c|ccccccc|}
 & a & b & c & d & e & f & g \\
\hline
u & +1 & 0 & 0 & 0 & +1 & 0 & 0 \\
v & 0 & -1 & +1 & 0 & -1 & -1 & 0 \\
w & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\
x & -1 & +1 & 0 & 0 & 0 & 0 & -1 \\
y & 0 & 0 & 0 & -1 & 0 & +1 & +1
\end{array}
$$

vertex-edge adj matrix

- Sum of rows $= 0$
- $n \times n$ submatrices have $\det = 0$

Counting spanning trees



$$\begin{array}{c@{}c}
 & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{array}{c} u \\ v \\ w \\ x \\ y \end{array} &
\begin{bmatrix}
+1 & 0 & 0 & 0 & +1 & 0 & 0 \\
0 & -1 & +1 & 0 & -1 & -1 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & +1 & +1
\end{bmatrix}
\end{array}$$

vertex-edge adj matrix

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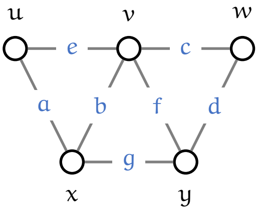
- $n \times n$ submatrices have $\det = 0$
- How about $(n-1) \times (n-1)$?

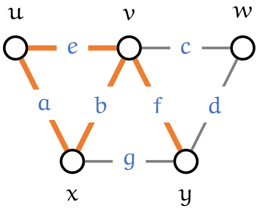
Counting spanning trees



$$\begin{array}{cc} & \begin{matrix} \;\;a & \;\;b & \;\;c & \;\;d & \;\;e & \;\;f & \;\;g \end{matrix} \\ \begin{matrix} u \\ v \\ w \\ x \\ y \end{matrix} & \begin{bmatrix} +1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 & +1 \end{bmatrix} \end{array}$$

vertex-edge adj matrix

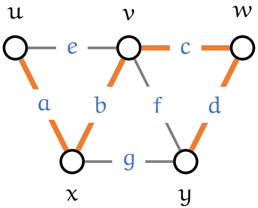
- Sum of rows $= 0$
- $n \times n$ submatrices have **det** $= 0$
- How about $(n-1) \times (n-1)$?
- If cycle exists, **det** $= 0$:



For some choice of signs:

$$\pm(\text{col } a) \pm (\text{col } b) \pm (\text{col } e) = 0$$

Otherwise, columns are a spanning tree. In this case $\mathbf{det} \in \{\pm 1\}$. Sketch:



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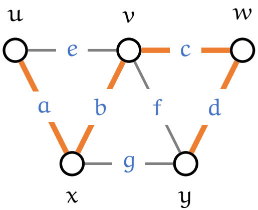

submatrix

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} u \\ v \\ w \\ x \end{array} & \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ -1 & +1 & 0 & 0 \end{bmatrix} \end{array}$$

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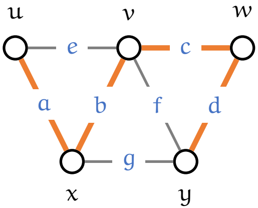

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added row u to x

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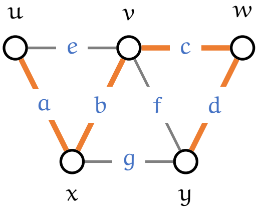

submatrix

$$\begin{array}{c|cccc|} & a & b & c & d \\ \hline u & +1 & 0 & 0 & 0 \\ v & 0 & -1 & +1 & 0 \\ w & 0 & 0 & -1 & +1 \\ x & -1 & +1 & 0 & 0 \end{array}$$

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added row x to v

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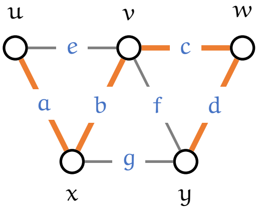

submatrix

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added row v to w

$$\begin{array}{c} \\ u \\ v \\ w \\ x \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 \end{bmatrix} \end{array}$$

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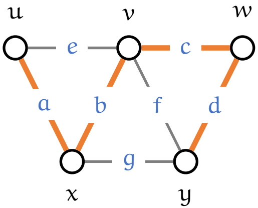

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permuted and fixed signs

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[Cauchy-Binet]

If A is $n \times m$ and B is $m \times n$:

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{\text{cols}=S}) \det(B_{\text{rows}=S}).$$

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- Next lecture: other determinant-based counting algs.