CS 263: Counting and Sampling

Nima Anari

Stanford University

slides for

Continuous Sampling

Review

Stochastic localization

For μ on $\mathbb{R}^n$, and adapted matrix process C_t, we define $\forall x$

$$d\mu_t(x) = \langle x - \mathsf{mean}(\mu_t), C_t\, dB_t \rangle \mu_t(x)$$

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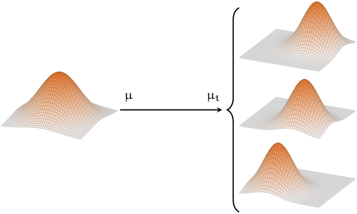
$$\mu_t \propto \mu \cdot \exp\left(-\frac{x^\intercal A_t x}{2} + \langle h_t, x \rangle\right)$$



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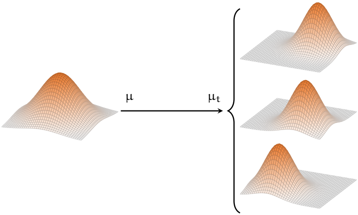

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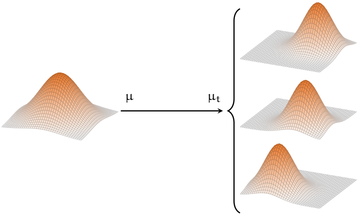

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$$\mathbb{E}\left[\mathrm{Ent}^{\phi}_{\mu_t}[f]\right] \geqslant \gamma \cdot \mathrm{Ent}^{\phi}_{\mu}[f]$$

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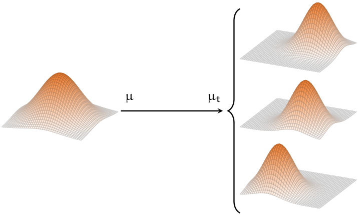

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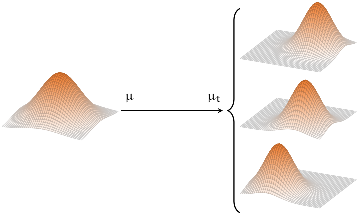

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[Eldan-Koehler-Zeitouni]

Glauber for Ising models μ on $\{\pm 1\}^n$

$$\mu(x) \propto \exp\left(\frac{x^\intercal J x}{2} + \langle h, x \rangle\right)$$

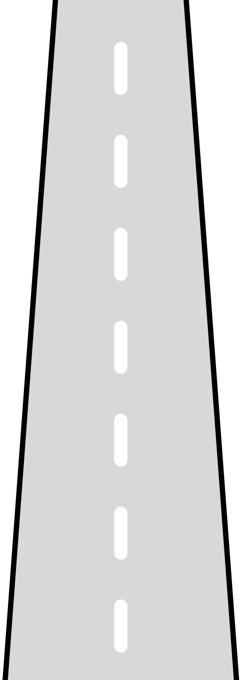
fast when $\lambda_{\max}(J) - \lambda_{\min}(J) < 1$.

Continuous Sampling

- Log-concave distributions
- Restricted Gaussian dynamics

Highlights

- Deterministic methods
- Markov chains



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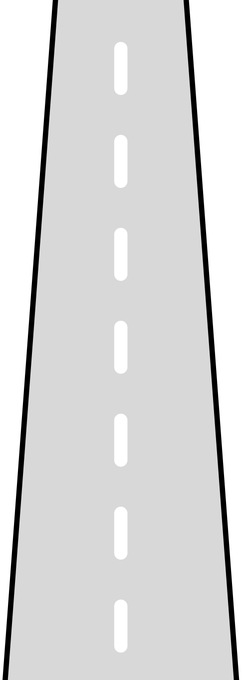
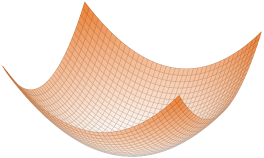


Continuous space

Optimization



$$U : \mathbb{R}^n \to \mathbb{R}$$

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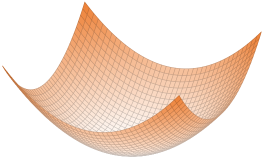

$$\mathfrak{U} : \mathbb{R}^n \to \mathbb{R}$$

- Tractable: $\mathfrak{U}$ convex

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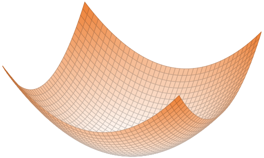

$$U : \mathbb{R}^n \to \mathbb{R}$$

- Tractable: U convex
- Even better (well-conditioned):

$$\alpha I \preceq \nabla^2 U \preceq \beta I.$$

and condition number $\kappa = \beta/\alpha$.
Gradient descent: $\mathrm{poly}(\kappa, \log(1/\epsilon))$.

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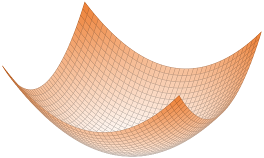

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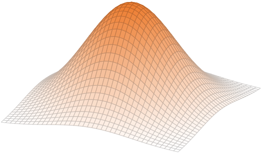

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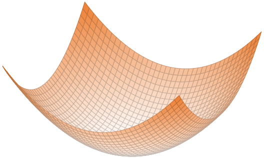

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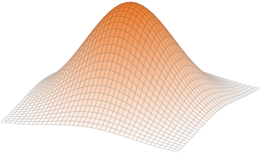

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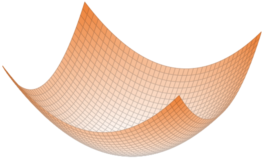

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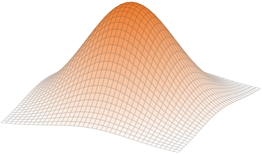

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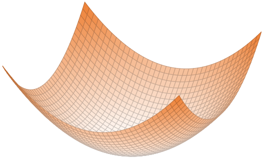

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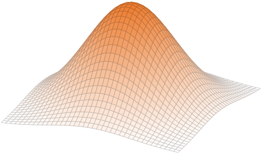

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- Best known: $\widetilde{O}(\sqrt{n}) \cdot \mathsf{poly}(\kappa)$ [Altschuler-Chewi]

We will see how to sample in $\mathsf{poly}(n, \kappa)$.

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Because $U(x) - \alpha\|x\|^2/2$ is convex and $U(x) - \beta\|x\|^2/2$ is concave.

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- Chance of acceptance:

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- This is $(\sqrt{2\pi\alpha}/\sqrt{2\pi\beta})^n = 1/\kappa^{n/2}$.

Rejection sampling works for $\kappa \leqslant 1 + \widetilde{O}(1/n)$

Restricted Gaussian dynamics [Lee-Shen-Tian]

Markov chain

- μ: dist on $\mathbb{R}^n$
- N: $x \mapsto y = x + g$ for $g \sim \mathcal{N}(0, I/\eta)$
- $P = NN^{\circ}$: then sample z w.p. $\propto$

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$\uparrow$

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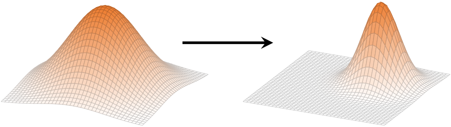

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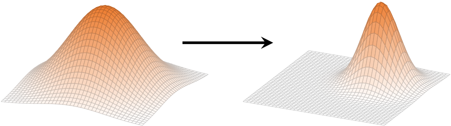

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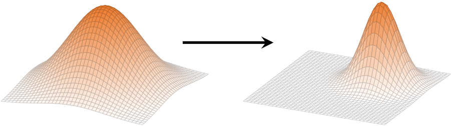

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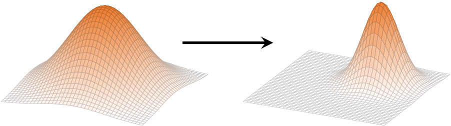

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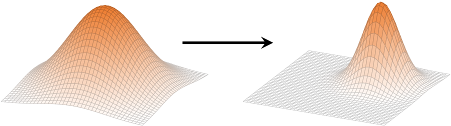

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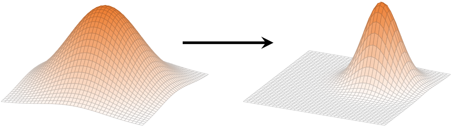

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- [Chen-Eldan]: same for entropy.

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- We will use $C_t = I$ today!

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- So, as long as we prove γ-approximate conservation of Ent^{ϕ}, we have proved N contracts Ent^{ϕ} by $1 - \gamma$. Do this at $t = \eta$. ☺

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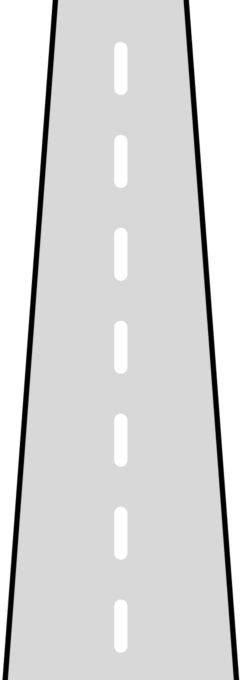
- This gives us $g(t) = 1/(\alpha + t)$, so
  $$\int_0^{\eta} g(t) dt = \log\left(\frac{\alpha + \eta}{\alpha}\right)$$
  which means $\gamma \geqslant \alpha/(\alpha + \eta)$. 🙂

Continuous Sampling

- Log-concave distributions
- Restricted Gaussian dynamics

Highlights

- Deterministic methods
- Markov chains



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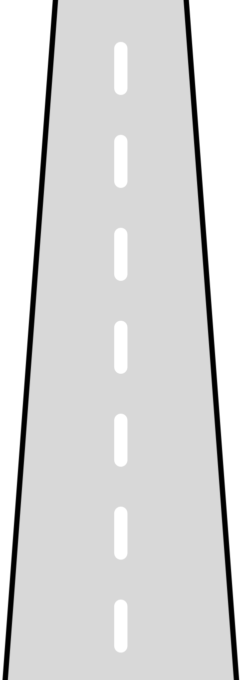
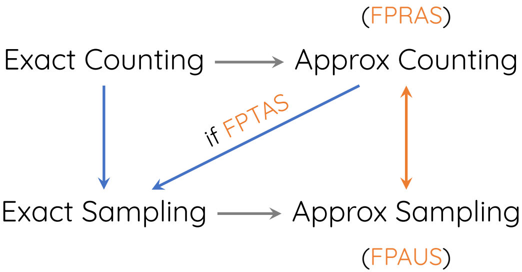


Theorem [Jerrum-Valiant-Vazirani]

For “self-reducible” problems:

approx counting ≡ approx sampling



(FPRAS)

Exact Counting

Approx Counting

if FPTAS

Exact Sampling

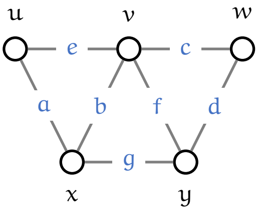
Approx Sampling

(FPAUS)

arrows are poly-time reductions

Deterministic counting

Determinant-based counting

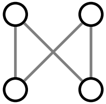


$$\begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline u & +1 & 0 & 0 & 0 & +1 & 0 & 0 \\ v & 0 & -1 & +1 & 0 & -1 & -1 & 0 \\ w & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ x & -1 & +1 & 0 & 0 & 0 & 0 & -1 \\ y & 0 & 0 & 0 & -1 & 0 & +1 & +1 \end{array}$$

$$\#\text{spanning trees} = \det(\text{matrix})$$

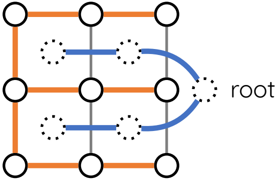
Laplacian, drop one row+col

- [Pólya]'s scheme:

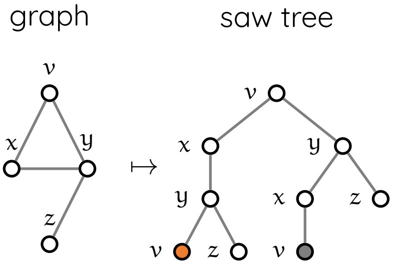


$$\det\left(\begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}\right) = \operatorname{per}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

- Good signing exists for planar graphs [Fisher-Kasteleyn-Temperley].



Correlation decay



- Root marginals are the same.
- Weak spatial mixing:

$$d_{TV}(\text{root} \mid \sigma, \text{root} \mid \sigma') \to 0$$

as the following goes to ∞:

$$\min\{d(\text{root}, u) \mid u \in S\}.$$

- Strong spatial mixing:

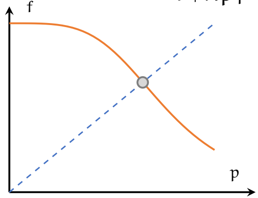
$$d_{TV}(\text{root} \mid \sigma, \text{root} \mid \sigma') \to 0$$

as the following goes to ∞:

$$\min\{d(\text{root}, u) \mid \sigma(u) \neq \sigma'(u)\}.$$

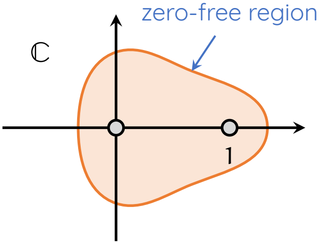
- [Weitz]'s alg forms truncated saw tree and uses recursion:

$$f(p_1, \ldots, p_d) = \frac{1}{1+\lambda p_1 \cdots p_d}$$

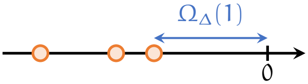


- Attractive exactly when $\lambda < \lambda_c(\Delta)$

[Barvinok]'s method



- [Barvinok]: approx $p(1)$ via $p^{(i)}(0)$ for $i = 0, \ldots, O(\log \deg(p))$
- Idea 1: Riemann map from disk
- Idea 2: trunc Taylor series of $\log p$
- Matchings via [Heilmann-Lieb]:



Markov chains

Transport

- Influence: X, X' differing in coord j:

$$d_{TV}\big(\mathrm{dist}(X_i \mid X_{-i}), \mathrm{dist}(X'_i \mid X'_{-i})\big)$$

- Call maximum value $\mathcal{J}[j \to i]$.

Dobrushin's condition

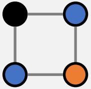
If columns of $\mathcal{J}$ sum to $\leqslant 1 - \delta$, then

$$\mathcal{W}(\nu P, \nu' P) \leqslant (1 - \delta/n)\, \mathcal{W}(\nu, \nu')$$

$$t_{\mathrm{mix}}(\epsilon) = O\big(\tfrac{1}{\delta} n \log(n/\epsilon)\big)$$

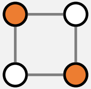
Example: coloring

- $\Omega = [q]^n$
- $\mathcal{J} \leqslant \frac{1}{q-\Delta} \cdot \mathrm{adj}$



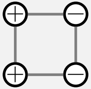
Example: hardcore

- $\Omega = \{0, 1\}^n$
- $\mathcal{J} \leqslant \frac{\lambda}{1+\lambda} \cdot \mathrm{adj}$



Example: Ising

- $\Omega = \{\pm 1\}^n$
- $\mathcal{J}[j \to i] \leqslant |\beta_{ij}|$



- Dobrushin++: if $c\,\mathcal{J} < (1 - \delta)c$

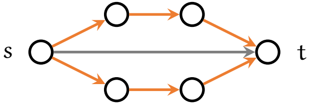
$$t_{\mathrm{mix}}(\epsilon) = O\left(\frac{n}{\delta} \log\left(\frac{n \cdot c_{\mathrm{max}}}{\epsilon \cdot c_{\mathrm{min}}}\right)\right)$$

- Existence: $\lambda_{\mathrm{max}}(\mathcal{J}) < 1$

Comparison

- P, P' reversible with same stationary distribution
- Comparison: route Q' through Q with low congestion and length.

$$\pi(\text{path} \mid X_0 = s, X_\ell = t)$$



Congestion

Suppose π is dist over paths and Q is ergodic flow. Congestion is

$$\max\left\{ \frac{\mathbb{P}_{\text{path}\sim\pi}[(x \to y) \in \text{path}]}{Q(x,y)} \;\middle|\; x \neq y \right\}$$

Lemma: comparison

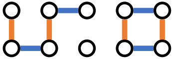
Suppose ρ, ρ' are χ^2 contraction rates:

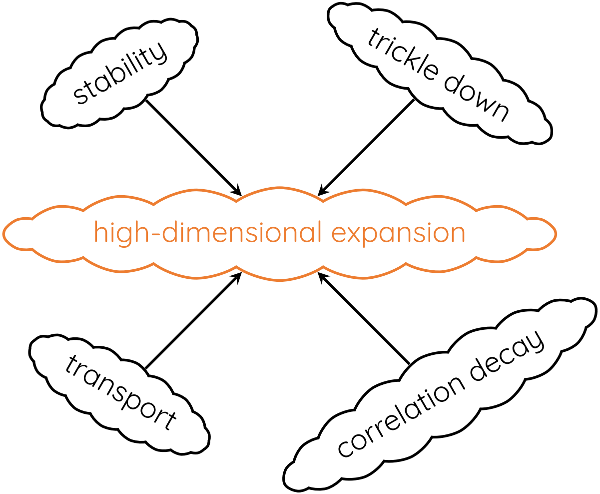
$$\rho \geqslant \frac{\rho'}{(\text{congestion}) \cdot (\text{max length})}$$

- If len $\leqslant 1$, can use any $\mathcal{D}_\phi$.
- Canonical paths: a few-to-one mapping enc from (s,t)-pairs whose path passes $x \to y$ to Ω:

$$\mu(s)\mu(t) \leqslant C \cdot \mu(\mathsf{enc}(s,t))Q(x,y)$$

- If M-to-one, then cong $\leqslant CM$.
- Matching walks mix in $\mathsf{poly}(n)$.





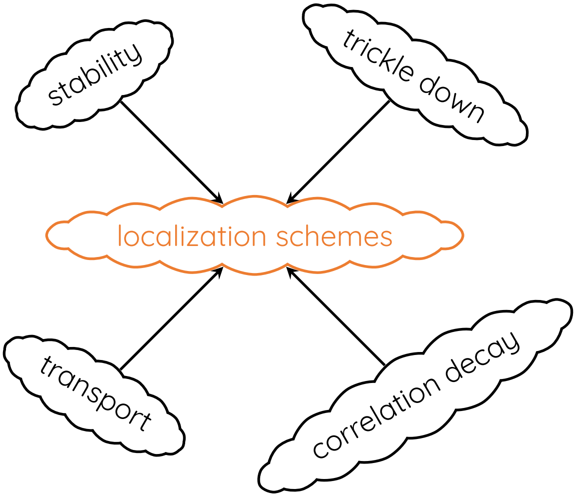
stability

trickle down

high-dimensional expansion

transport

correlation decay



stability

trickle down

localization schemes

transport

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