CS 263: Counting and Sampling

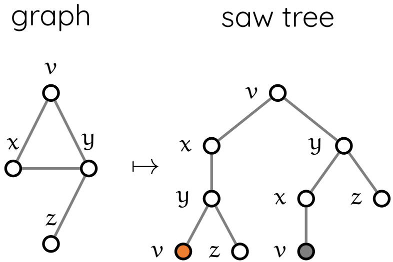
Nima Anari

Stanford University

slides for

Universality of HDX

Review



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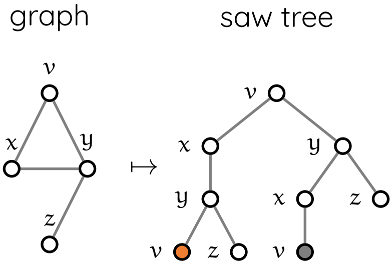

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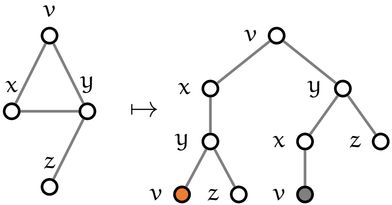
graph saw tree



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- Weak spatial mixing:

$$d_{\mathrm{TV}}(\text{root} \mid \sigma, \text{root} \mid \sigma') \to 0$$

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$$\min\{d(\text{root}, u) \mid u \in S\}.$$

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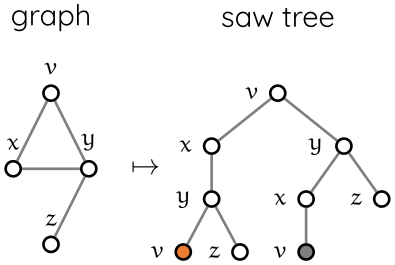

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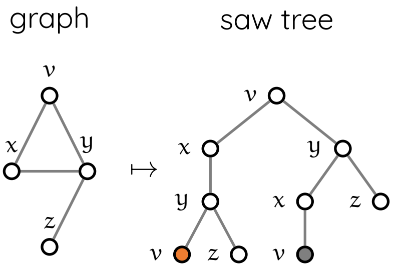

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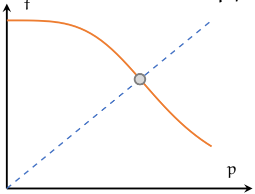
$$f(p_1, \dots, p_d) = \frac{1}{1+\lambda p_1 \cdots p_d}$$



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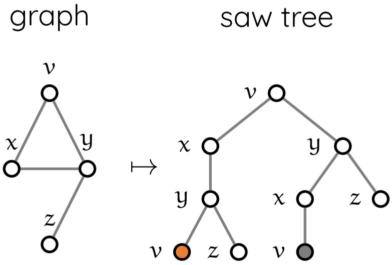
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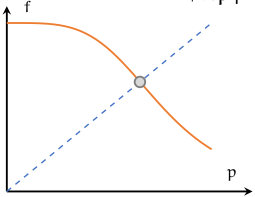

- Attractive exactly when $\lambda < \lambda_c(\Delta)$

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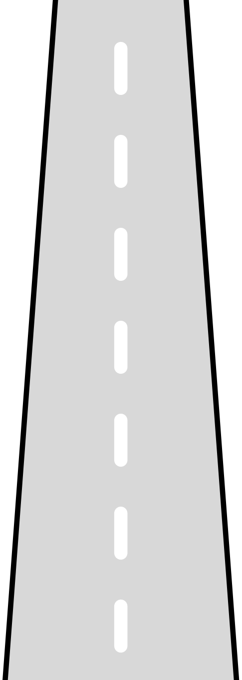
- Open: runtime of deterministic algs seem to be $n^{O(\log \Delta)}$, can we remove bad dependency on Δ?

HDX via Correlation Decay

- Influences
- Fast sampling

HDX via Transport

- Universality



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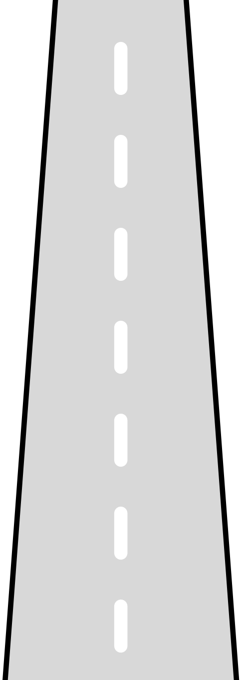

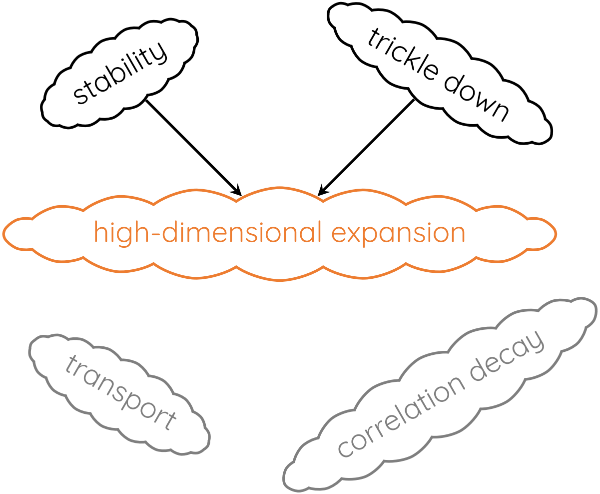

stability

trickle down

high-dimensional expansion

transport

correlation decay

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- Equal because $r' = rh$ and $s' = sh$.

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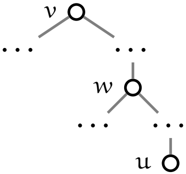

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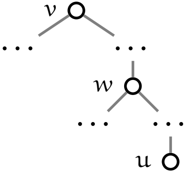

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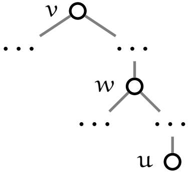

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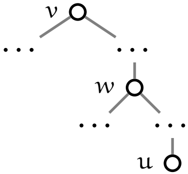

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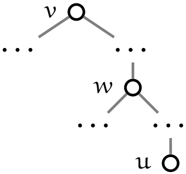

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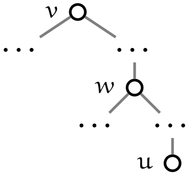

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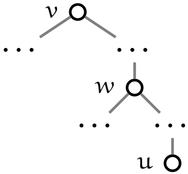

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- If $\|\nabla f\|_1 \leqslant 1 - \epsilon$, we'd be done.
- Unfortunately, this is not the case. But we can use previous trick

$$g = \psi \circ f(\psi^{-1}(\cdot), \dots, \psi^{-1}(\cdot))$$

and g will have $\|\nabla g\|_1 \leqslant 1 - \epsilon$ everywhere for hardcore.

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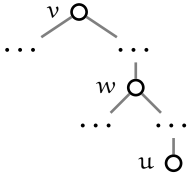

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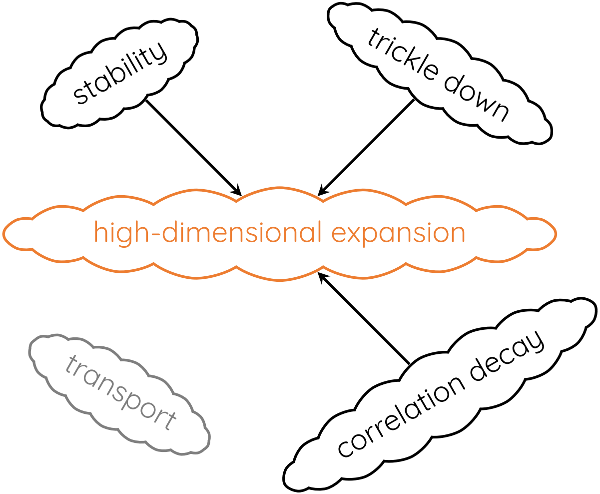
- If we track $t_\ell = \sum_{u \in \text{level } \ell} |\mathcal{I}[v \to u]|$, and show for some $\ell = O(1)$, we have $t_\ell \leqslant 1 - \epsilon$, we are done. ☺
- Let p_u be $\mathbb{P}[X_u = 0]$ in u's subtree.
- Using $q_u = \log((1 - p_u)/p_u)$, we have recursion
$$q_v = f(q_{u_1}, \ldots, q_{u_k})$$
where $u_1, \ldots, u_k$ are children of v.
- Claim:
$$\mathcal{I}[v \to u_i] = \partial_i f(q_{u_1}, \ldots, q_{u_k})$$
- If $\|\nabla f\|_1 \leqslant 1 - \epsilon$, we'd be done.
- Unfortunately, this is not the case. But we can use previous trick
$$g = \psi \circ f(\psi^{-1}(\cdot), \ldots, \psi^{-1}(\cdot))$$
and g will have $\|\nabla g\|_1 \leqslant 1 - \epsilon$ everywhere for hardcore.
- Derivatives of ψ are "nice". Contraction of f after $O(1)$ levels.



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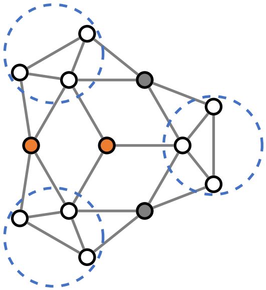
- If $n - \ell = \delta n$ for sufficiently small δ, cond on ℓ random verts, we get islands of size $\simeq O_{\Delta,\lambda}(\log n)$. 🙂



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loose bound

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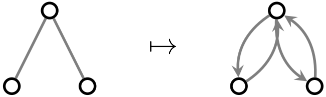

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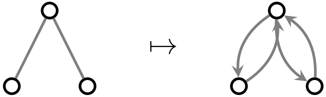

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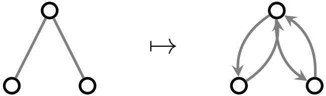

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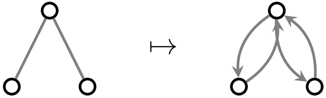

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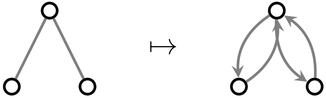

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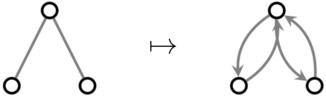

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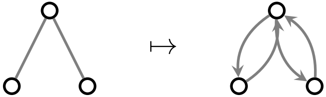

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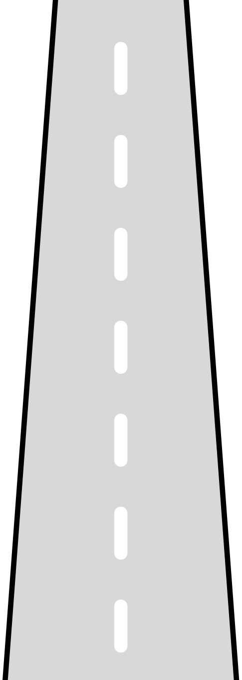
HDX via Transport

- Universality



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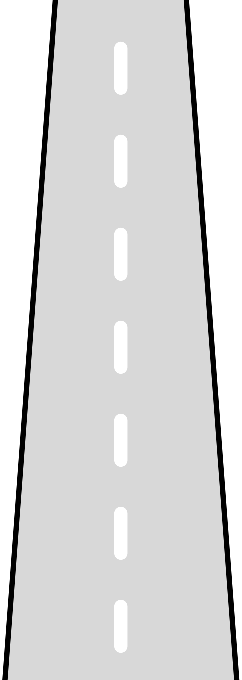

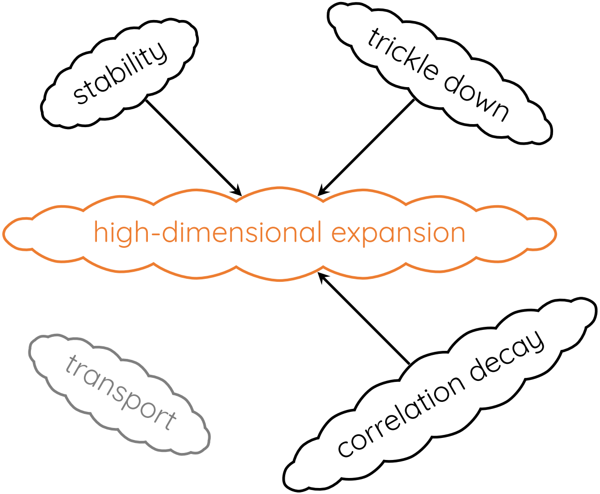

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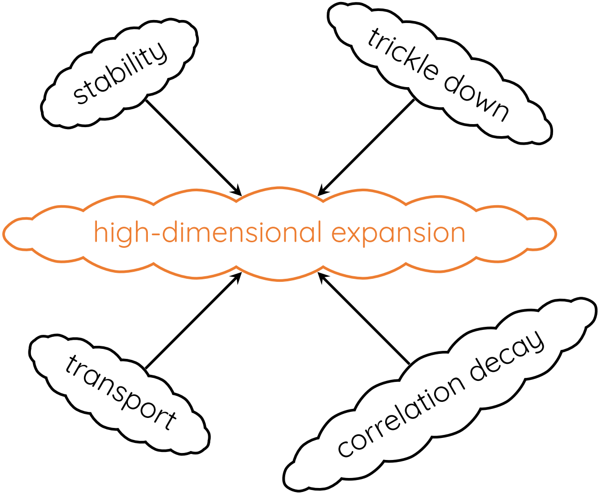

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