CS 263: Counting and Sampling

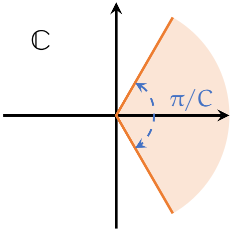
Nima Anari

Stanford University

slides for

Correlation Decay

Review



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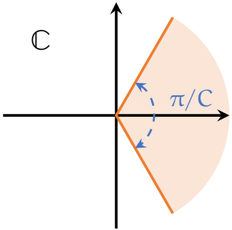

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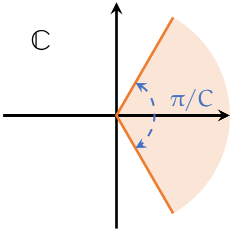

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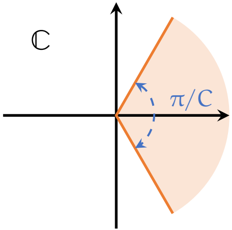

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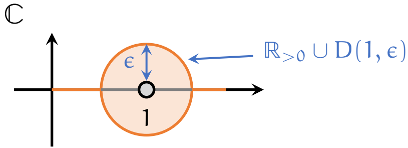
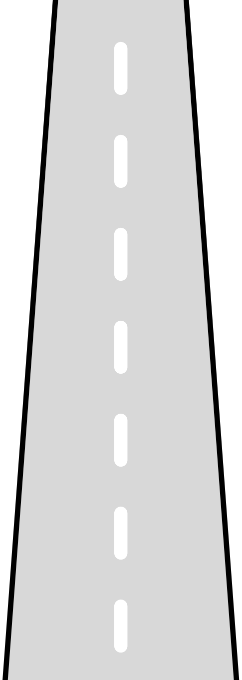


Correlation Decay

- Marginals on trees
- Self-avoiding walk tree
- Weak and strong spatial mixing

HDX via Correlation Decay

- Tree vs graph polynomials



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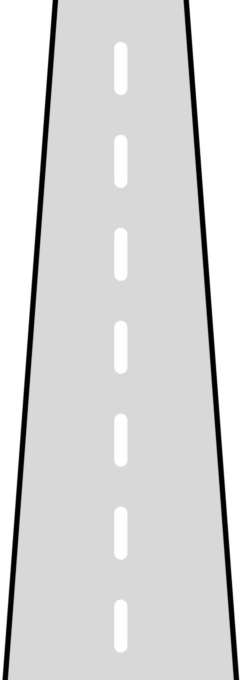

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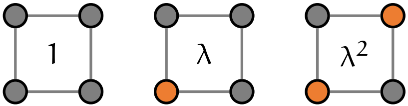

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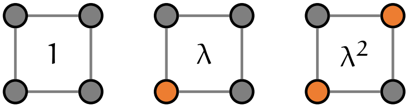

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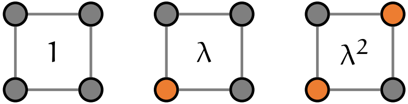

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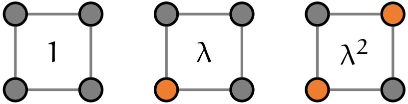

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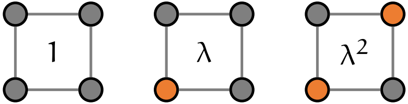

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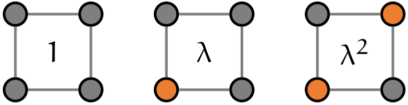

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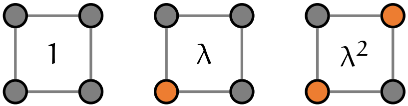

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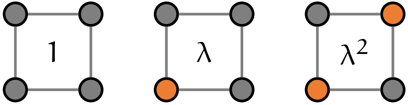

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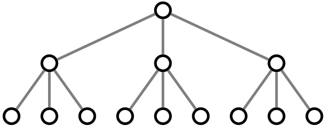

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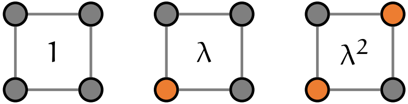

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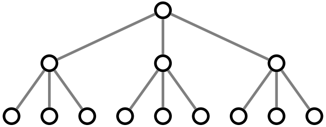

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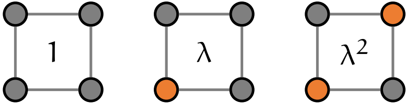

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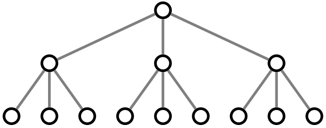

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Correlation decay

For any two configs σ, σ' for leaves

$$d_{TV}(\text{root} \mid \sigma, \text{root} \mid \sigma') \leqslant f(\text{height})$$

where $f(\text{height}) \to 0$ as height $\to \infty$.

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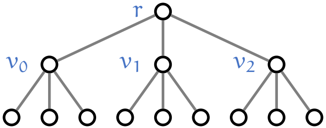
- Note: this generalizes to other "repulsive" two-spin systems.
- [Weitz]'s algorithm estimates marginals $\mathbb{P}[X_v]$ via local computation.
- Multiplying estimated marginals approximates partition function $\sum_{\text{ind } S} \lambda^S \simeq$

$$\frac{1}{\mathbb{P}[X_{v_1} = 0] \cdot \mathbb{P}[X_{v_2} = 0 \mid X_{v_1} = 0] \cdots \mathbb{P}[X_{v_n} = 0 \mid X_{v_1} = \cdots = X_{v_{n-1}} = 0]}$$

Marginals on trees

- On trees, (conditional) marginals can be computed recursively.



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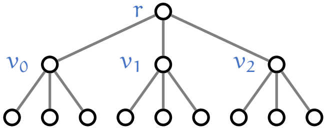

Recursive computation

If $p_v = \mathbb{P}[X_v = 0]$ in v's subtree, and $d = \Delta - 1$:

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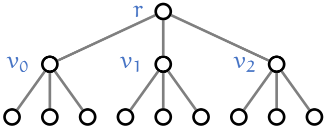

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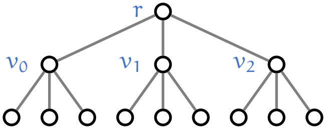

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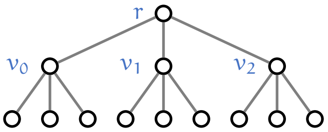

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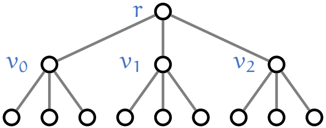

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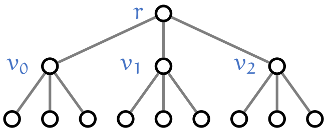

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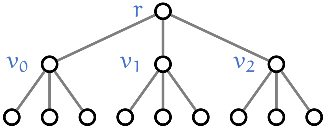

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- Combining all gives

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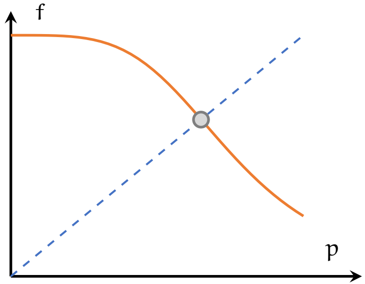

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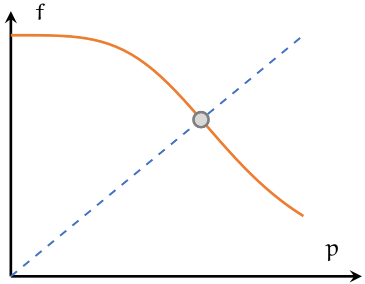

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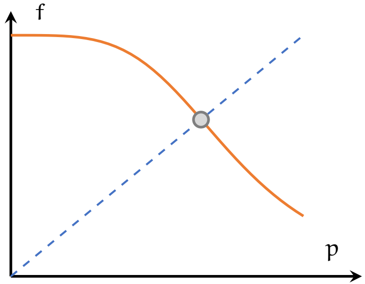

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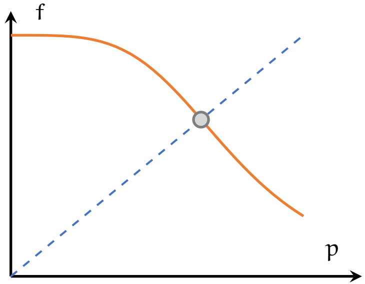

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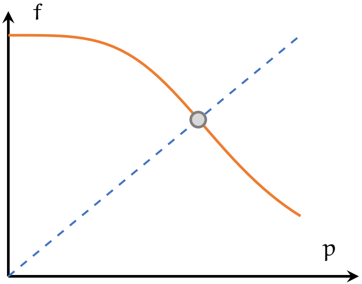

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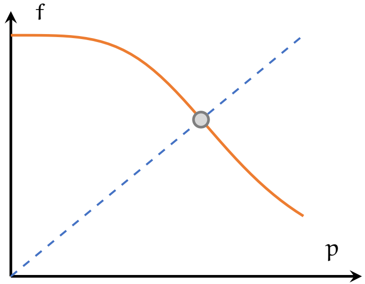

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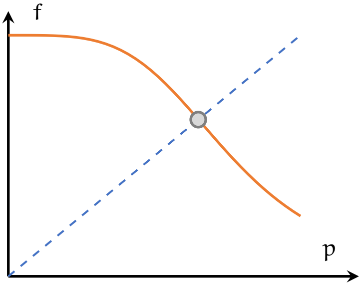

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- When $\lambda > \lambda_c$ we have $|f'(p^*)| > 1$. In this case, p^* is repulsive.
- When attractive, for $p_0 \in$ basin of attraction:

  $$f^{\text{height}}(p_0) \to p^*$$

  Luckily for this f, basin of attraction is all of $[0, 1]$.

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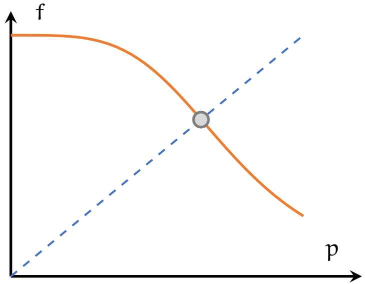

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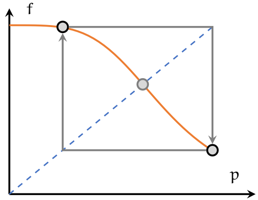
- When $\lambda < \lambda_c$ we have $|f'(p^*)| < 1$. In this case, p^* is attractive.
- When $\lambda > \lambda_c$ we have $|f'(p^*)| > 1$. In this case, p^* is repulsive.
- When attractive, for $p_0 \in$ basin of attraction:

  $$f^{\text{height}}(p_0) \to p^*$$

  Luckily for this f, basin of attraction is all of $[0, 1]$.
- When repulsive, oscillating:



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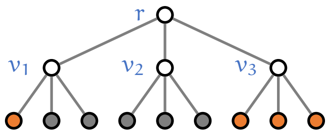

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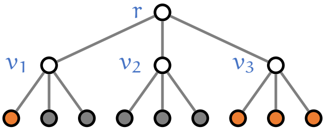

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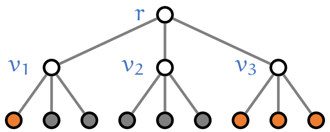

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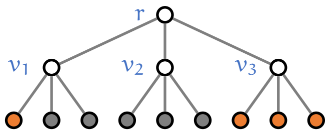

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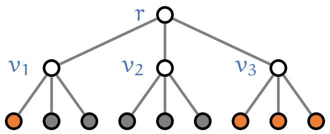

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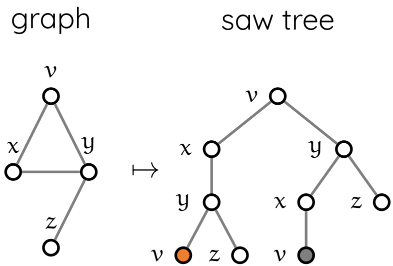
- Correlation decay on trees exactly below λ_c. 🙂

What about general graphs?

Self-avoiding walk tree



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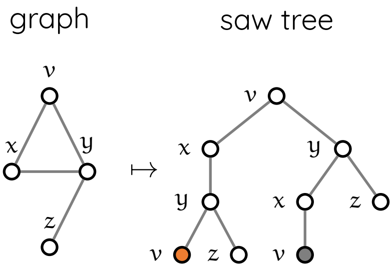

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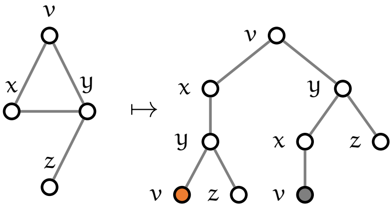
graph saw tree



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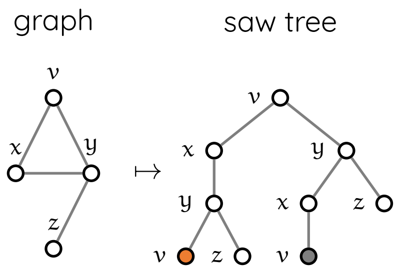

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Root marginal same on self-avoiding walk tree and graph.

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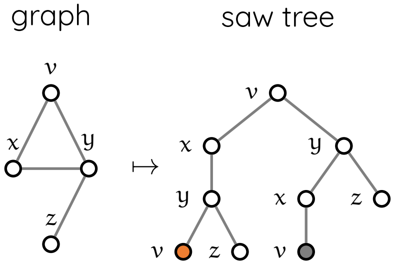

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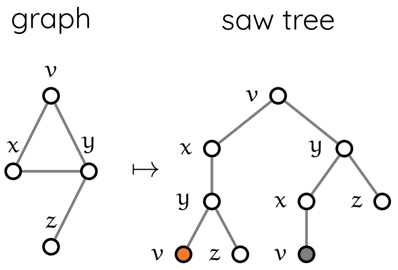

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- Correlation decay allows us to cut the tree height by conditioning nodes at level $\simeq \log n$ arbitrarily.

Spatial mixing

Let us fix root, and condition subset $S \subseteq V$ of verts two ways: $\sigma, \sigma' \subseteq \{0,1\}^S$.

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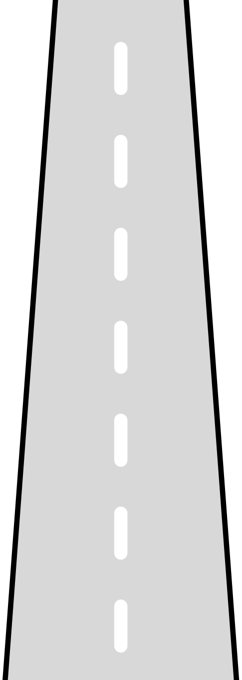
Need to show tree $\equiv$ graph ...

Correlation Decay

- Marginals on trees
- Self-avoiding walk tree
- Weak and strong spatial mixing

HDX via Correlation Decay

- Tree vs graph polynomials



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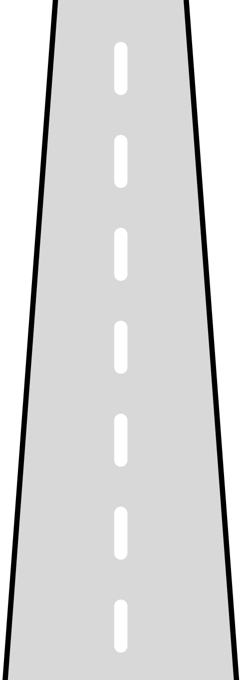

Polynomials

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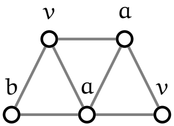

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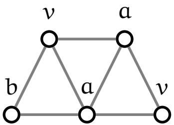

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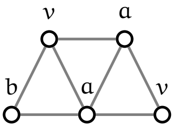

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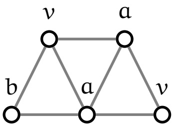

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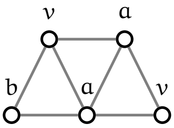

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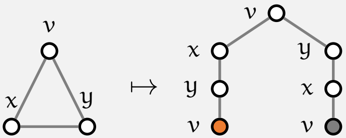

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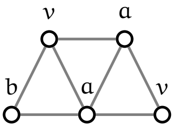

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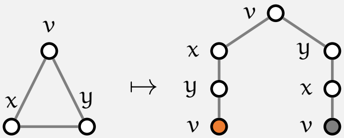

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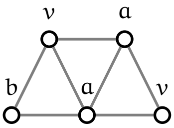

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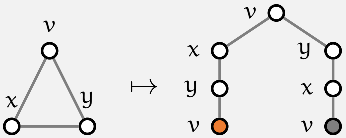
Lemma [Godsil, . . .]

Form the saw tree (inherit labels). There is a polynomial h such that

$$g_{\text{tree}} = g_{\text{graph}} \cdot h$$

Example

Hardcore model with $\lambda = 1$:



- $g_{\text{graph}} = 1 + z_v + z_x + z_y$
- $g_{\text{tree}} = (1 + z_v + z_x + z_y)(1 + z_x)$

Corollary

Root marginals same on tree/graph.

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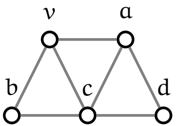
- Assume no univariate/vertex factors in μ for simplicity (e.g., $\lambda = 1$). They can be added later by change of variable $z_v \mapsto \lambda z_v$:



graph $G = ([n], E)$

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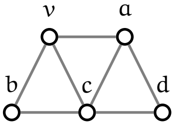

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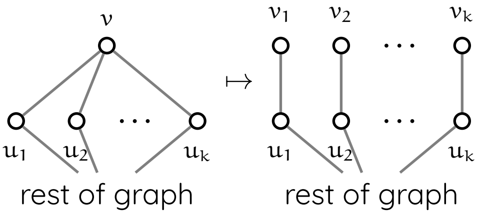
- We will prove via induction . . .

▷ Split root v:


v
u_1 u_2 $\cdots$ u_k
rest of graph
$\mapsto$
v_1 v_2 $\cdots$ v_k
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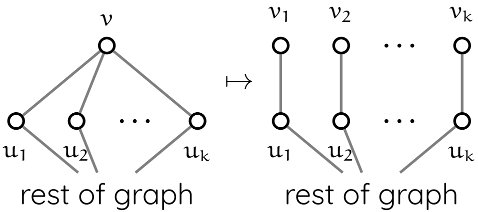
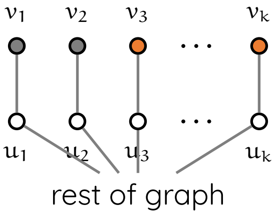


- Pinnings: pin $v_1, \ldots, v_i$ to 0 and $v_{i+1}, \ldots, v_k$ to 1. Let poly be g_i.



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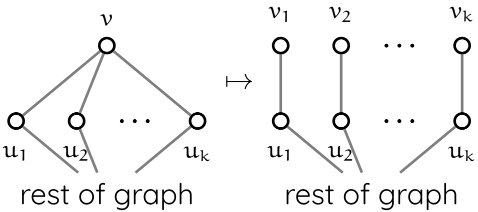

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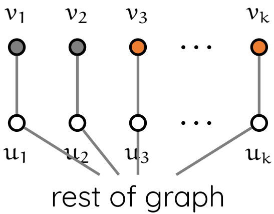

- We have

$$g_{\text{graph}} = z_v g_0 + g_k$$

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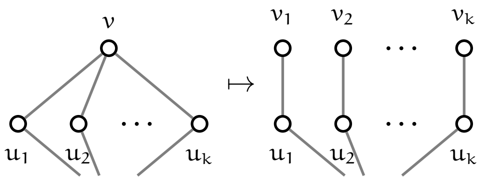

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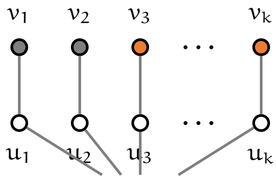

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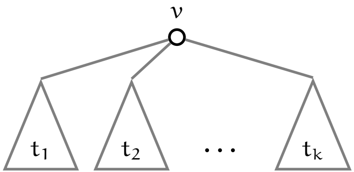
- Now looking at tree



$$g_{\text{tree}} = z_v \prod_i r_i + \prod_i s_i$$

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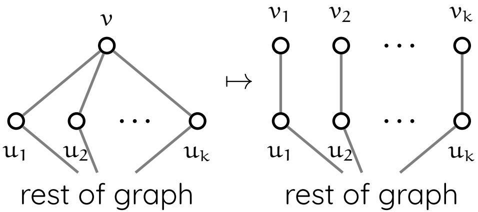

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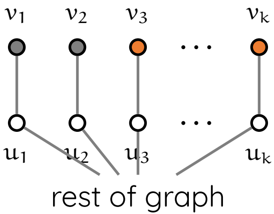

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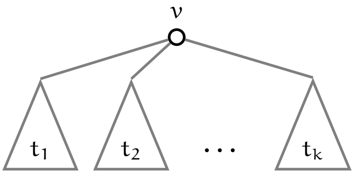
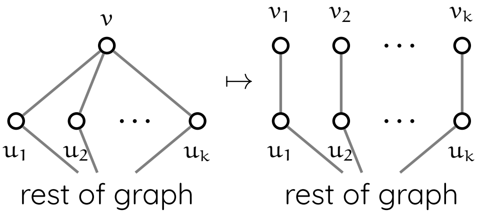


$$g_{\text{tree}} = z_v \prod_i r_i + \prod_i s_i$$

- By induction we have for each i
$$z_v r_i + s_i = h_i \cdot (z_v g_{i-1} + g_i)$$
i.e., $r_i = h_i g_{i-1}$ and $s_i = h_i g_i$.

- Split root v:



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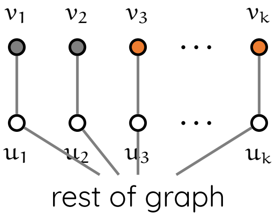

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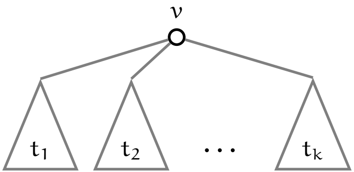

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i.e., $r_i = h_i g_{i-1}$ and $s_i = h_i g_i$.

- Now let $h = g_1 \cdots g_{k-1} \cdot h_1 \cdots h_k$:

$$h g_{\text{graph}} = z_v \prod_i r_i + \prod_i s_i$$

This finishes proof of [Weitz].