CS 263: Counting and Sampling

Nima Anari

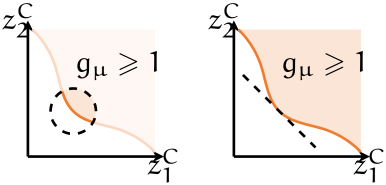
Stanford University

slides for

Stability

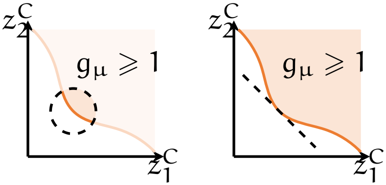
Review

- C-SI: $\log g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[C]{z}) \leqslant$ tangent at $\mathbb{1}$



Review

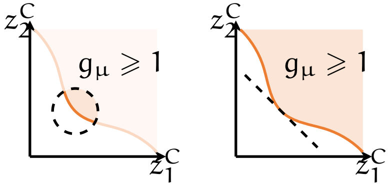
- C-SI: $\log g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[C]{z}) \leqslant$ tangent at $\mathbb{1}$



- C-FLC: $\log g_\mu(\sqrt[C]{z})$ concave on $\mathbb{R}^n_{>0}$

Review

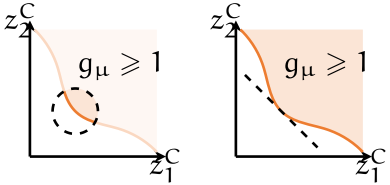
- C-SI: $\log g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[C]{z}) \leqslant$ tangent at $\mathbb{1}$



- C-FLC: $\log g_\mu(\sqrt[C]{z})$ concave on $\mathbb{R}^n_{>0}$
- C-SI under all exp tilts $\implies$ C-EI

Review

- C-SI: $\log g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[C]{z}) \leqslant$ tangent at $\mathbb{1}$

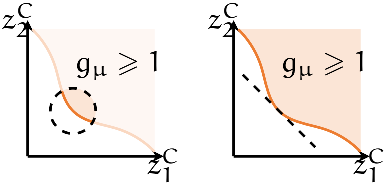


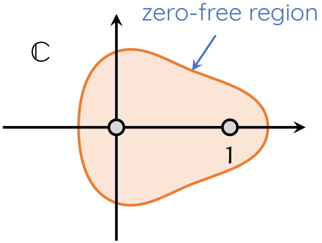
- C-FLC: $\log g_\mu(\sqrt[C]{z})$ concave on $\mathbb{R}^n_{>0}$
- C-SI under all exp tilts $\implies$ C-EI
- Down-up on matroids:

$$t_{\text{mix}} \leqslant O(k \log k + k \log \log n)$$

Review

- C-SI: $\log g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[C]{z}) \leqslant$ tangent at $\mathbb{1}$





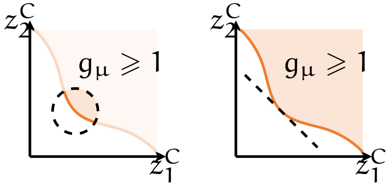
- [Barvinok]: approx $p(1)$ via $p^{(i)}(0)$ for $i = 0, \ldots, O(\log \deg(p))$

- C-FLC: $\log g_\mu(\sqrt[C]{z})$ concave on $\mathbb{R}^n_{>0}$
- C-SI under all exp tilts $\implies$ C-EI
- Down-up on matroids:

$$t_{\text{mix}} \leqslant O(k \log k + k \log \log n)$$

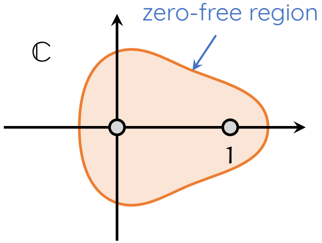
Review

- C-SI: $\log g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[C]{z}) \leqslant$ tangent at $\mathbb{1}$



- C-FLC: $\log g_\mu(\sqrt[C]{z})$ concave on $\mathbb{R}^n_{>0}$
- C-SI under all exp tilts $\implies$ C-EI
- Down-up on matroids:

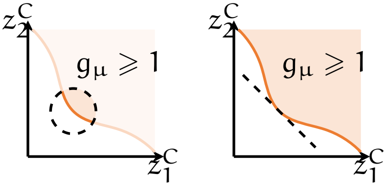
$$t_{\text{mix}} \leqslant O(k \log k + k \log\log n)$$



- [Barvinok]: approx $p(1)$ via $p^{(i)}(0)$ for $i = 0, \ldots, O(\log \deg(p))$
- Idea 1: Riemann map from disk

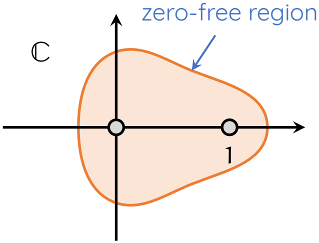
Review

- C-SI: $\log g_\mu(\sqrt[c]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[c]{z}) \leqslant$ tangent at $\mathbb{1}$



- C-FLC: $\log g_\mu(\sqrt[c]{z})$ concave on $\mathbb{R}^n_{>0}$
- C-SI under all exp tilts $\implies$ C-EI
- Down-up on matroids:

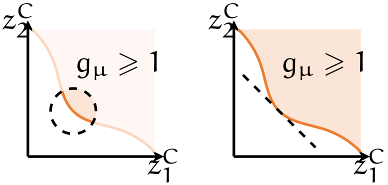
$$t_{\text{mix}} \leqslant O(k \log k + k \log \log n)$$



- [Barvinok]: approx $p(1)$ via $p^{(i)}(0)$ for $i = 0, \ldots, O(\log \deg(p))$
- Idea 1: Riemann map from disk
- Idea 2: trunc Taylor series of $\log p$

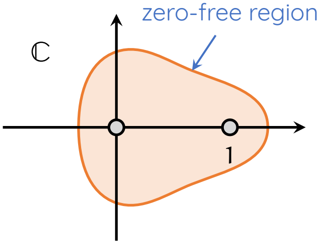
Review

- C-SI: $\log g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$
- C-EI: $\log g_\mu(\sqrt[C]{z}) \leqslant$ tangent at $\mathbb{1}$

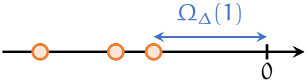


- C-FLC: $\log g_\mu(\sqrt[C]{z})$ concave on $\mathbb{R}^n_{>0}$
- C-SI under all exp tilts $\implies$ C-EI
- Down-up on matroids:

$$t_{mix} \leqslant O(k \log k + k \log\log n)$$

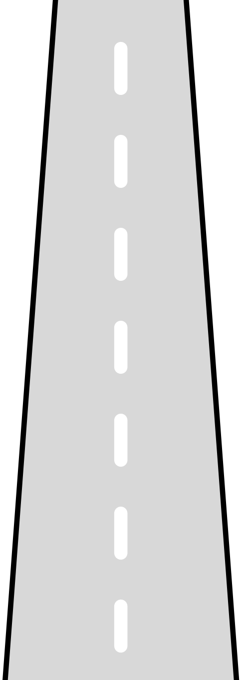


- [Barvinok]: approx $p(1)$ via $p^{(i)}(0)$ for $i = 0, \ldots, O(\log \deg(p))$
- Idea 1: Riemann map from disk
- Idea 2: trunc Taylor series of $\log p$
- Matchings via [Heilmann-Lieb]:



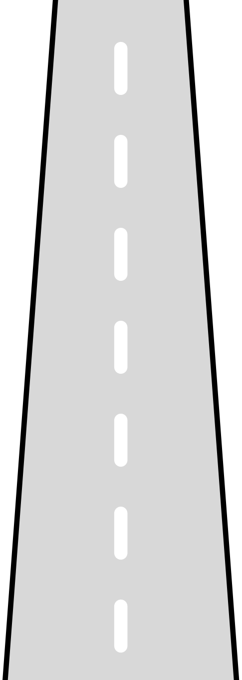
HDX via Stability

- Sector stability
- Half-plane stability
- Schwarz lemma



HDX via Stability

- Sector stability
- Half-plane stability
- Schwarz lemma



Multivariate stability

Stability

Poly g is U-stable for $U \subseteq \mathbb{C}$ when

$$z_1, \ldots, z_n \in U \implies g(z_1, \ldots, z_n) \neq 0$$

Multivariate stability

Stability

Poly g is $\mathbb{U}$-stable for $\mathbb{U} \subseteq \mathbb{C}$ when

$$z_1, \ldots, z_n \in \mathbb{U} \implies g(z_1, \ldots, z_n) \neq 0$$

Example: det point process

For vectors $v_1, \ldots, v_n \in \mathbb{R}^k$, let

$$\mu(S) \propto \det\left([v_i]_{i \in S}\right)^2.$$

Then g_μ is half-plane-stable.

e.g., $\{z \mid \mathrm{Re}(z) > 0\}$

Multivariate stability

Stability

Poly g is U-stable for $U \subseteq \mathbb{C}$ when

$$z_1, \ldots, z_n \in U \implies g(z_1, \ldots, z_n) \neq 0$$

Example: det point process

For vectors $v_1, \ldots, v_n \in \mathbb{R}^k$, let

$$\mu(S) \propto \det\left([v_i]_{i \in S}\right)^2.$$

Then g_μ is half-plane-stable.

e.g., $\{z \mid \mathrm{Re}(z) > 0\}$

Proof:

Multivariate stability

Stability

Poly g is U-stable for $U \subseteq \mathbb{C}$ when

$$z_1, \ldots, z_n \in U \implies g(z_1, \ldots, z_n) \neq 0$$

Example: det point process

For vectors $v_1, \ldots, v_n \in \mathbb{R}^k$, let

$$\mu(S) \propto \det\left([v_i]_{i \in S}\right)^2.$$

Then g_μ is half-plane-stable.

e.g., $\{z \mid \mathrm{Re}(z) > 0\}$

Proof:

- For $A_i = v_i v_i^T \succeq 0$, we have

$$g_\mu = \det(z_1 A_1 + \cdots + z_n A_n)$$

Multivariate stability

Stability

Poly g is U-stable for $U \subseteq \mathbb{C}$ when

$$z_1, \ldots, z_n \in U \implies g(z_1, \ldots, z_n) \neq 0$$

Example: det point process

For vectors $v_1, \ldots, v_n \in \mathbb{R}^k$, let

$$\mu(S) \propto \det\left([v_i]_{i \in S}\right)^2.$$

Then g_μ is half-plane-stable.

e.g., $\{z \mid \mathrm{Re}(z) > 0\}$

Proof:

- For $A_i = v_i v_i^T \succeq 0$, we have
$$g_\mu = \det(z_1 A_1 + \cdots + z_n A_n)$$

- If we let
$$B = \sum_i \mathrm{Re}(z_i) A_i, \; C = \sum_i \mathrm{Im}(z_i) A_i$$
then $B \succ 0$ and C is sym. Roots of $\det(B + xC)$ are real and $\neq \sqrt{-1}$.

Multivariate stability

Stability

Poly g is U-stable for $U \subseteq \mathbb{C}$ when

$$z_1, \ldots, z_n \in U \implies g(z_1, \ldots, z_n) \neq 0$$

Example: det point process

For vectors $v_1, \ldots, v_n \in \mathbb{R}^k$, let

$$\mu(S) \propto \det\left([v_i]_{i \in S}\right)^2.$$

Then g_μ is half-plane-stable.

e.g., $\{z \mid \mathrm{Re}(z) > 0\}$

Proof:

- For $A_i = v_i v_i^T \succeq 0$, we have

$$g_\mu = \det(z_1 A_1 + \cdots + z_n A_n)$$

- If we let

$$B = \sum_i \mathrm{Re}(z_i) A_i, C = \sum_i \mathrm{Im}(z_i) A_i$$

then $B \succ 0$ and C is sym. Roots of $\det(B + xC)$ are real and $\neq \sqrt{-1}$.

Example: monomers [Heilmann-Lieb]

$$\sum_{\text{matchings}} \left(\prod_{i \text{ matched}} z_i \right)$$

is $\{z \mid \mathrm{Re}(z) > 0\}$-stable.

Multivariate stability

Stability

Poly g is U-stable for $U \subseteq \mathbb{C}$ when

$$z_1, \dots, z_n \in U \implies g(z_1, \dots, z_n) \neq 0$$

Example: det point process

For vectors $v_1, \dots, v_n \in \mathbb{R}^k$, let

$$\mu(S) \propto \det\left([v_i]_{i \in S}\right)^2.$$

Then g_μ is half-plane-stable.

e.g., $\{z \mid \text{Re}(z) > 0\}$

Proof:

- For $A_i = v_i v_i^T \succeq 0$, we have
$$g_\mu = \det(z_1 A_1 + \cdots + z_n A_n)$$

- If we let
$$B = \sum_i \text{Re}(z_i) A_i, C = \sum_i \text{Im}(z_i) A_i$$
then $B \succ 0$ and C is sym. Roots of $\det(B + xC)$ are real and $\neq \sqrt{-1}$.

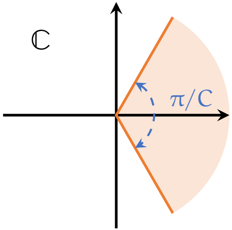
Example: monomers [Heilmann-Lieb]

$$\sum_{\text{matchings}} \left(\prod_{i \text{ matched}} z_i \right)$$

is $\{z \mid \text{Re}(z) > 0\}$-stable.

- Exercise: prove this via induction similar to univariate case.

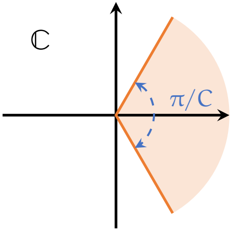
Sector stability



Polynomial $g(z_1, \ldots, z_n)$ is C sector stable if for all $z_1, \ldots, z_n \in$ sector

$$g(z_1, \ldots, z_n) \neq 0.$$

Sector stability

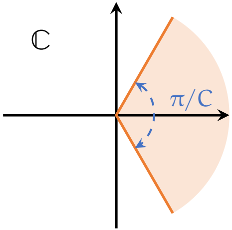


Polynomial $g(z_1, \dots, z_n)$ is C sector stable if for all $z_1, \dots, z_n \in$ sector

$$g(z_1, \dots, z_n) \neq 0.$$

- g_{DPP} is hom and 1 sector stable.

Sector stability

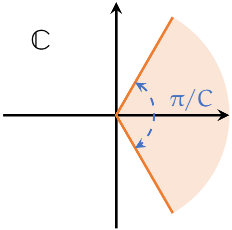


Polynomial $g(z_1, \dots, z_n)$ is C sector stable if for all $z_1, \dots, z_n \in$ sector

$$g(z_1, \dots, z_n) \neq 0.$$

- g_{DPP} is hom and 1 sector stable.
- The matching poly is NOT hom.

Sector stability



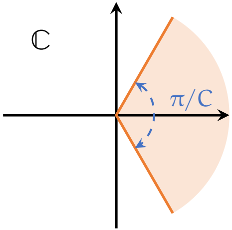
Polynomial $g(z_1, \dots, z_n)$ is C sector stable if for all $z_1, \dots, z_n \in$ sector

$$g(z_1, \dots, z_n) \neq 0.$$

- g_{DPP} is hom and 1 sector stable.
- The matching poly is NOT hom.
- Homogenization:

$$g(z_1, \dots, z_n) \mapsto y_1 \cdots y_n g\left(\frac{z_1}{y_1}, \dots, \frac{z_n}{y_n}\right)$$

Sector stability



Polynomial $g(z_1, \ldots, z_n)$ is C sector stable if for all $z_1, \ldots, z_n \in$ sector

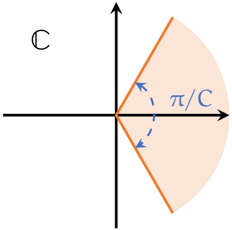
$$g(z_1, \ldots, z_n) \neq 0.$$

- g_{DPP} is hom and 1 sector stable.
- The matching poly is NOT hom.
- Homogenization:

$$g(z_1, \ldots, z_n) \mapsto y_1 \cdots y_n g\left(\frac{z_1}{y_1}, \ldots, \frac{z_n}{y_n}\right)$$

- The homogenized matching poly is 2 sector stable.

Sector stability



Polynomial $g(z_1, \ldots, z_n)$ is C sector stable if for all $z_1, \ldots, z_n \in$ sector

$$g(z_1, \ldots, z_n) \neq 0.$$

- g_{DPP} is hom and 1 sector stable.
- The matching poly is NOT hom.
- Homogenization:

$$g(z_1, \ldots, z_n) \mapsto y_1 \cdots y_n g\left(\frac{z_1}{y_1}, \ldots, \frac{z_n}{y_n}\right)$$

- The homogenized matching poly is 2 sector stable.
- We will show certain forms of stability imply HDX and thus mixing of random walks.

HDX via stability

g is generating poly of μ on $\binom{[n]}{k}$:

[Gårding]

Half-plane stable $\implies$

$$\log g(z_1, \ldots, z_n) \text{ concave}$$

HDX via stability

g is generating poly of μ on $\binom{[n]}{k}$:

[Gårding]

Half-plane stable $\implies$

$$\log g(z_1, \ldots, z_n) \text{ concave}$$

[Alimohammadi-A-Shiragur-Vuong]

C sector stable $\implies$

$$\log g(\sqrt[2C]{z_1}, \ldots, \sqrt[2C]{z_n}) \text{ concave}$$

HDX via stability

g is generating poly of μ on $\binom{[n]}{k}$:

[Gårding]

Half-plane stable $\implies$

$$\log g(z_1, \dots, z_n) \text{ concave}$$

[Alimohammadi-A-Shiragur-Vuong]

C sector stable $\implies$

$$\log g(\sqrt[2C]{z_1}, \dots, \sqrt[2C]{z_n}) \text{ concave}$$

- The extra 2 can be dropped for homogenized matchings.

HDX via stability

g is generating poly of μ on $\binom{[n]}{k}$:

[Gårding]

Half-plane stable $\implies$

$$\log g(z_1, \ldots, z_n) \text{ concave}$$

[Alimohammadi-A-Shiragur-Vuong]

C sector stable $\implies$

$$\log g(\sqrt[2C]{z_1}, \ldots, \sqrt[2C]{z_n}) \text{ concave}$$

- The extra 2 can be dropped for homogenized matchings.

- We get $(2C)$-SI, $(2C)$-EI, and $\text{poly}(k)$ mixing when $C = O(1)$.

HDX via stability

g is generating poly of μ on $\binom{[n]}{k}$:

[Gårding]

Half-plane stable $\implies$

$$\log g(z_1, \ldots, z_n) \text{ concave}$$

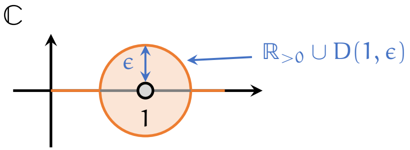
[Alimohammadi-A-Shiragur-Vuong]

C sector stable $\implies$

$$\log g(\sqrt[2C]{z_1}, \ldots, \sqrt[2C]{z_n}) \text{ concave}$$

- The extra 2 can be dropped for homogenized matchings.

- We get $(2C)$-SI, $(2C)$-EI, and **poly**(k) mixing when $C = O(1)$.
- To just get SI, entire sector is not needed. Sufficient region:



HDX via stability

g is generating poly of μ on $\binom{[n]}{k}$:

[Gårding]

Half-plane stable $\implies$

$$\log g(z_1, \ldots, z_n) \text{ concave}$$

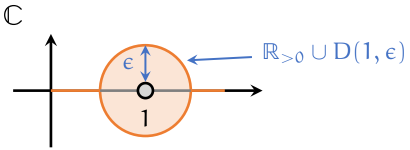
[Alimohammadi-A-Shiragur-Vuong]

C sector stable $\implies$

$$\log g(\sqrt[2C]{z_1}, \ldots, \sqrt[2C]{z_n}) \text{ concave}$$

- The extra 2 can be dropped for homogenized matchings.

- We get $(2C)$-SI, $(2C)$-EI, and $\mathbf{poly}(k)$ mixing when $C = O(1)$.
- To just get SI, entire sector is not needed. Sufficient region:



[Chen-Liu-Vigoda]

If μ originates from product space, handles can be removed with extra assumptions (lower and/or upper bounds) on marginals.

Half-plane-stability to log-concavity

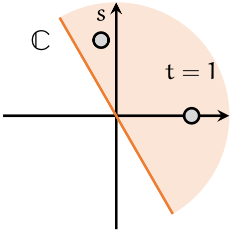
Sketch of [Gårding'51] (1 sector stable):

1. To show g is log-concave on $\mathbb{R}^n_{>0}$, enough to consider restriction to 2-dim subspaces. For $u, v \in \mathbb{R}^n_{>0}$: $h(s,t) = g(su + tv)$.

Half-plane-stability to log-concavity

Sketch of [Gårding'51] (1 sector stable):

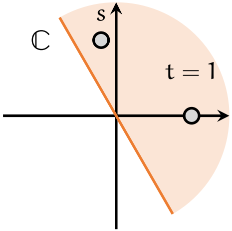
1. To show g is log-concave on $\mathbb{R}^n_{>0}$, enough to consider restriction to 2-dim subspaces. For $u, v \in \mathbb{R}^n_{>0}$: $h(s,t) = g(su + tv)$.
2. $h(s, 1)$ must have roots $\in \mathbb{R}_{\leqslant 0}$.



Half-plane-stability to log-concavity

Sketch of [Gårding'51] (1 sector stable):

1. To show g is log-concave on $\mathbb{R}^n_{>0}$, enough to consider restriction to 2-dim subspaces. For $u, v \in \mathbb{R}^n_{>0}$: $h(s,t) = g(su + tv)$.
2. $h(s,1)$ must have roots $\in \mathbb{R}_{\leqslant 0}$.



3. $h(s,t) = \prod_i (a_i s + b_i t)$ for $a_i, b_i \in \mathbb{R}_{\geqslant 0}$. This implies log-concavity.

Sketch of [Alimohammadi-A-Shiragur-Vuong] (C sector stable):

1. Let us bound the ℓ_1 norm of i-th row of correlation matrix Ψ. Fact:

$$\lambda_{\max}(\Psi) \leqslant \max\{\ell_1(\text{row } i) \mid i\}.$$

2. Let $\mathbb{1}_S$ be the indicator of S sampled from μ. Then, there is a vector $w \in \{\pm 1\}^n$ for which

$$\ell_1(\text{row i}) = \mathbb{E}[\langle w, \mathbb{1}_S\rangle \mid i \in S] - \mathbb{E}[\langle w, \mathbb{1}_S\rangle].$$

3. We show that conditioning on $i \in S$ changes $\mathbb{E}[\langle w, \mathbb{1}_S\rangle]$ by at most $2C$.
4. The following "polynomial" is still sector-stable

$$\mathbb{E}[z^{\langle w, \mathbb{1}_S\rangle}] \propto g(z, z^{-1}, \dots).$$

5. By scaling z_i with positive reals, any positive combination ($\alpha, \beta > 0$) below remains sector-stable:

$$\alpha \cdot \mathbb{E}[z^{\langle w, \mathbb{1}_S\rangle} \mid i \in S] + \beta \cdot \mathbb{E}[z^{\langle w, \mathbb{1}_S\rangle} \mid i \notin S]$$

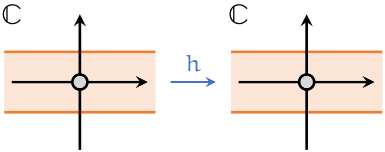
6 The ratio avoids negative reals when $z \in$ sector:

$$\frac{\mathbb{E}[z^{\langle w, \mathbb{1}_S \rangle} \mid i \in S]}{\mathbb{E}[z^{\langle w, \mathbb{1}_S \rangle} \mid i \notin S]}$$

7 There is a complex-analytic branch of log defined.

$$f(z) := \log\left(\frac{\mathbb{E}[z^{\langle w, \mathbb{1}_S \rangle} \mid i \in S]}{\mathbb{E}[z^{\langle w, \mathbb{1}_S \rangle} \mid i \notin S]}\right)$$

$$h(y) = f(e^{y/2C}) - f(1).$$



8 Derivative of h at $y = 0$ is bounded by 1 (by Shwarz's lemma):

$$\frac{d}{dy} h \bigg|_{y=0} = \frac{\mathbb{E}[\langle w, \mathbb{1}_S \rangle \mid i \in S] - \mathbb{E}[\langle w, \mathbb{1}_S \rangle \mid i \notin S]}{2C}.$$

Schwarz lemma

Schwarz lemma

If ϕ is a holomorphic map from $D(0,1)$ to $D(0,1)$ and $\phi(0) = 0$, then

$$|\phi'(0)| \leqslant 1.$$

Schwarz lemma

Schwarz lemma

If ϕ is a holomorphic map from $D(0,1)$ to $D(0,1)$ and $\phi(0) = 0$, then

$$|\phi'(0)| \leqslant 1.$$

Sketch of proof:

Schwarz lemma

Schwarz lemma

If ϕ is a holomorphic map from $\mathbb{D}(0,1)$ to $\mathbb{D}(0,1)$ and $\phi(0) = 0$, then

$$|\phi'(0)| \leqslant 1.$$

Sketch of proof:

- Maximum principle: any holomorphic map achieves maximum of $|\cdot|$ on boundary.

Schwarz lemma

Schwarz lemma

If ϕ is a holomorphic map from $D(0,1)$ to $D(0,1)$ and $\phi(0) = 0$, then

$$|\phi'(0)| \leqslant 1.$$

Sketch of proof:

- Maximum principle: any holomorphic map achieves maximum of $|\cdot|$ on boundary.
- Apply this to $\phi(z)/z$.

Schwarz lemma

Schwarz lemma

If ϕ is a holomorphic map from $D(0, 1)$ to $D(0, 1)$ and $\phi(0) = 0$, then

$$|\phi'(0)| \leqslant 1.$$

Sketch of proof:

- Maximum principle: any holomorphic map achieves maximum of $|\cdot|$ on boundary.
- Apply this to $\phi(z)/z$.
- On the boundary $|\phi(z)/z| \leqslant 1$. The value at 0 is $\phi'(0)$.

Schwarz lemma

Schwarz lemma

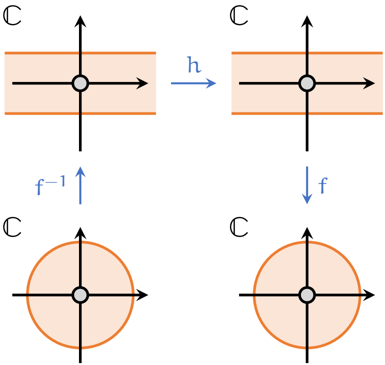
If ϕ is a holomorphic map from $D(0,1)$ to $D(0,1)$ and $\phi(0) = 0$, then

$$|\phi'(0)| \leqslant 1.$$

Sketch of proof:

- Maximum principle: any holomorphic map achieves maximum of $|\cdot|$ on boundary.
- Apply this to $\phi(z)/z$.
- On the boundary $|\phi(z)/z| \leqslant 1$. The value at 0 is $\phi'(0)$.

- Consider $f(z) = \frac{e^{z/2}-1}{e^{z/2}+1}$:



Schwarz lemma

Schwarz lemma

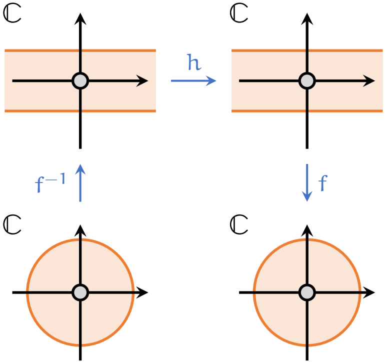
If ϕ is a holomorphic map from $D(0,1)$ to $D(0,1)$ and $\phi(0)=0$, then

$$|\phi'(0)| \leqslant 1.$$

Sketch of proof:

- Maximum principle: any holomorphic map achieves maximum of $|\cdot|$ on boundary.
- Apply this to $\phi(z)/z$.
- On the boundary $|\phi(z)/z| \leqslant 1$. The value at 0 is $\phi'(0)$.

- Consider $f(z) = \frac{e^{z/2}-1}{e^{z/2}+1}$:



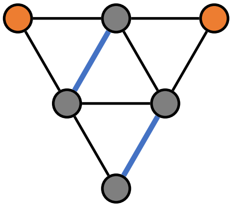
- Apply Schwarz to $\phi = f \circ h \circ f^{-1}$:

$$\left| f'(0) \cdot h'(0) \cdot \frac{1}{f'(0)} \right| \leqslant 1.$$

Monomer walks

$$\mu(S \times \{\textcolor{orange}{\bullet}\} \cup \overline{S} \times \{\textcolor{gray}{\bullet}\}) = \sum \{\text{momomer-dimer weights} \mid \text{monomers} = S\}$$

$$\mu : \binom{\text{vertices} \times \{\textcolor{gray}{\bullet}, \textcolor{orange}{\bullet}\}}{|\text{vertices}|} \to \mathbb{R}_{\geqslant 0}$$

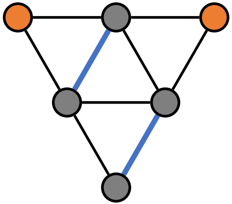


By [Heilmann-Lieb], g_μ is 2 sector stable.

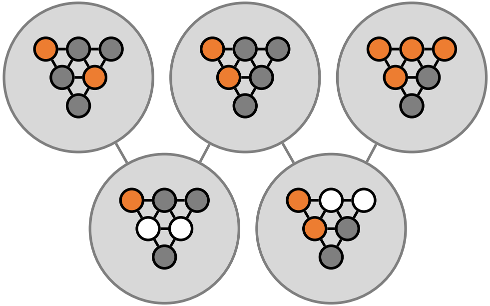
Monomer walks

$$\mu(S \times \{{\color{orange}\bullet}\} \cup \overline{S} \times \{{\color{gray}\bullet}\}) = \sum \{\text{momomer-dimer weights} \mid \text{monomers} = S\}$$

$$\mu : \binom{\text{vertices} \times \{{\color{gray}\bullet}, {\color{orange}\bullet}\}}{|\text{vertices}|} \to \mathbb{R}_{\geqslant 0}$$



By [Heilmann-Lieb], g_μ is 2 sector stable.
Down-up walk/block dynamics:



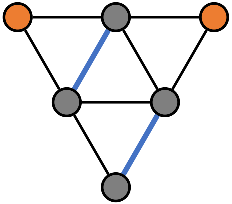
Corollary

Mixing in $\widetilde{O}(|\text{vertices}|^2)$.

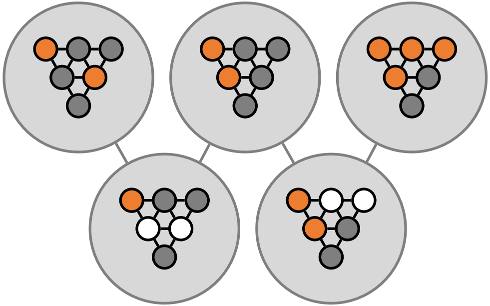
Monomer walks

$$\mu(S \times \{{\color{orange}\bullet}\} \cup \overline{S} \times \{{\color{gray}\bullet}\}) = \sum\{\text{momomer-dimer weights} \mid \text{monomers} = S\}$$

$$\mu : \binom{\text{vertices} \times \{{\color{gray}\bullet}, {\color{orange}\bullet}\}}{|\text{vertices}|} \to \mathbb{R}_{\geqslant 0}$$



By [Heilmann-Lieb], g_μ is 2 sector stable.
Down-up walk/block dynamics:

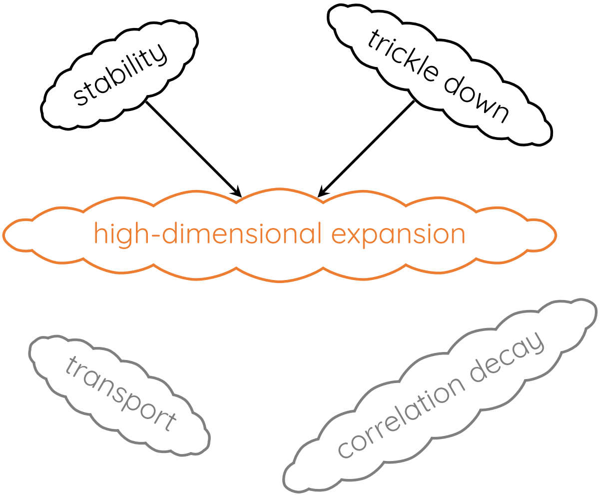


Corollary

Mixing in $\widetilde{O}(|\text{vertices}|^2)$.

Application: planar graphs

Sample from monomer-dimer systems on planar graphs in $\mathsf{poly}(n)$ time.


stability
trickle down
high-dimensional expansion
transport
correlation decay