CS 263: Counting and Sampling

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Stanford University

slides for

Zeros of Polynomials

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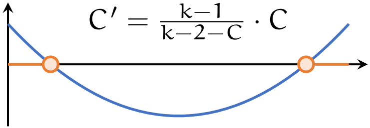

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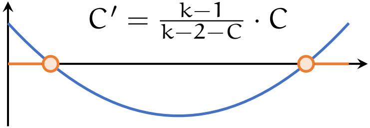

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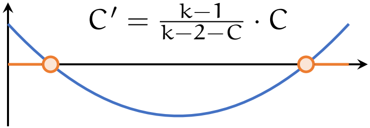

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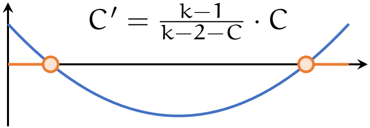

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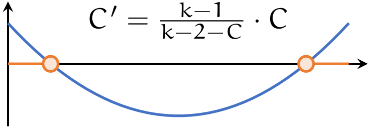

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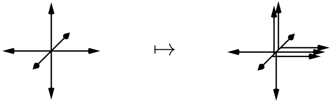

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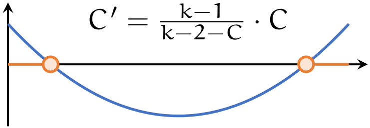

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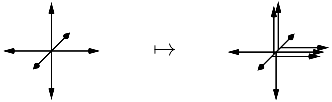
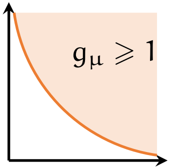


- Exp tilts of matroids are 1-SI:

$$\nabla^2 \log g \preceq 0 \text{ on } \mathbb{R}^n_{>0}$$



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Polynomial Interpolation

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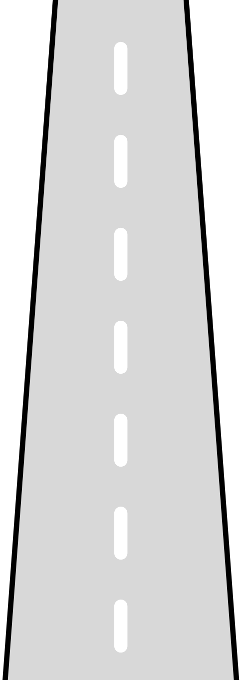
- Taylor approximation
- Riemann mappings



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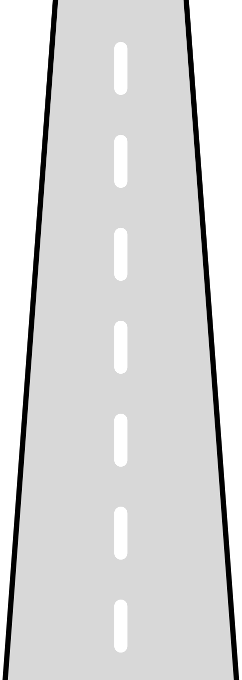

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For all distributions ν,

$$\mathcal{D}_{\mathsf{KL}}(\nu D_{k \to 1} \parallel \mu D_{k \to 1}) \leqslant \frac{C}{k} \cdot \mathcal{D}_{\mathsf{KL}}(\nu \parallel \mu).$$

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- If $q = (q_1, \dots, q_n)$ is some distribution on $[n] = \binom{[n]}{1}$, then
$$\inf\{\mathcal{D}_{\mathrm{KL}}(\nu \parallel \mu) \mid \nu D_{k\to 1} = q\} = -\log\left(\inf_{z_1,\dots,z_n>0} \frac{\sum_S \mu(S)\prod_{i\in S} z_i}{z_1^{kq_1}\cdots z_n^{kq_n}}\right).$$

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- This finishes the proof. 🙂

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- If $y_i = z_i^C$, this is the same as
$$g_\mu(\sqrt[C]{y_i})^{C/k} \leqslant \underbrace{\langle \mu D_{k\to 1}, y\rangle}_{\text{linear tangent at } \mathbb{1}}$$

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$$g_\mu(z) \leqslant h(z)$$
- If $y_i = z_i^C$, this is the same as
$$g_\mu(\sqrt[C]{y_i})^{C/k} \leqslant \underbrace{\langle \mu D_{k\to 1}, y\rangle}_{\text{linear tangent at } \mathbb{1}}$$
- For $f(z) = g(\sqrt[C]{z})^{C/k}$, we have $\nabla f(\mathbb{1}) = \mu D_{k\to 1}$ and $\nabla^2 f(\mathbb{1}) \propto$
$$\mathrm{cov} - C\cdot\left(\mathrm{diag}(\mathrm{mean}) - \frac{\mathrm{mean}\,\mathrm{mean}^\intercal}{k}\right)$$

Folklore lemma

For a d-homogeneous function f, tfae:

1. $\{z \mid f(z) \geqslant 1\}$ convex
2. $\sqrt[d]{f}$ is concave
3. $\log f$ is concave

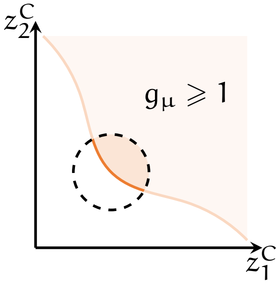
Similarly tfae:

1. $\{z \mid f(z) \geqslant 1\} \subseteq \{z \mid \langle \frac{\nabla f(\mathbb{1})}{d f(\mathbb{1})}, z\rangle \geqslant 1\}$
2. $\sqrt[d]{f}$ bounded by tangent at $\mathbb{1}$
3. $\log f$ bounded by tangent at $\mathbb{1}$

Polynomial view of HDX

Spectral Independence

Level sets of $g_\mu(\sqrt[C]{z})$ locally convex at $\mathbb{1}$.



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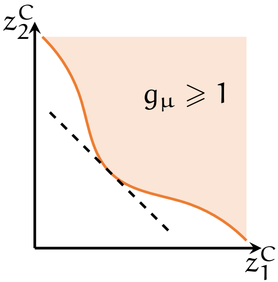

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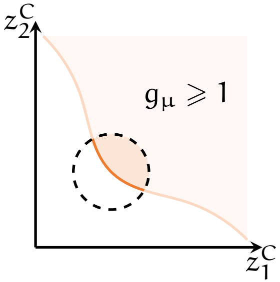

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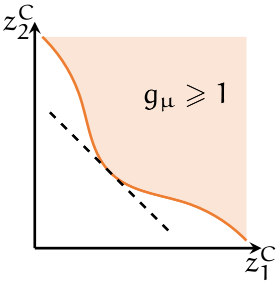

Theorem [A-Jain-Koehler-Pham-Vuong'21]

C-spectral independence for all exp tilts $\Longrightarrow$ C-entropic independence.

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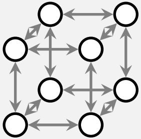
Example: hypercube

- $\{0,1\}^n \hookrightarrow \binom{[2n]}{n}$
- Glauber becomes $D_{n\to n-1}$



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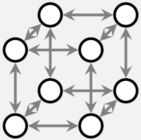

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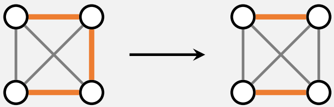

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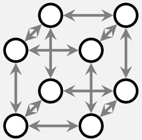

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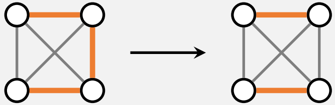

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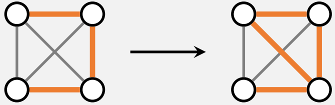
Example: spanning trees (II)



- P: add edge u.a.r., then drop
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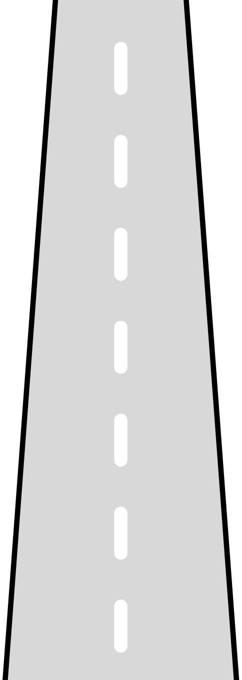
Entropic Independence

- Fractional log-concavity

Polynomial Interpolation

- Matching polynomial
- Taylor approximation
- Riemann mappings



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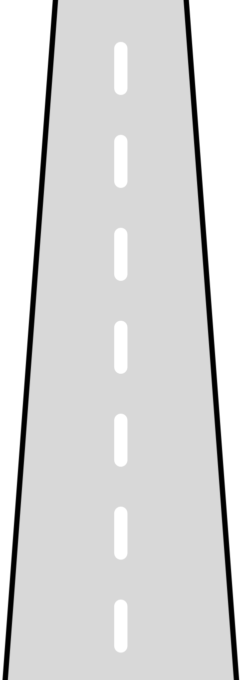

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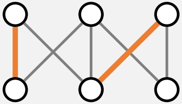

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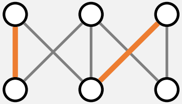

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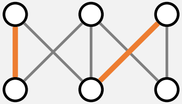

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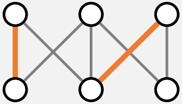

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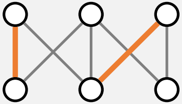

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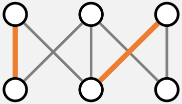

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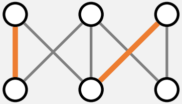

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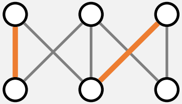

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For $\Delta = O(1)$, we can multiplicatively approximate $p_G(1)$ using
$$p_G^{(0)}(0), \ldots, p_G^{O(\log n)}(0)$$

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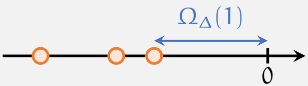
Zeros/roots of p_G are real. In fact they lie in $(-\infty, -\Omega_\Delta(1)]$.



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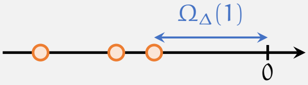

- Roots are singularities of $\log p_G$. Derivatives of $\log p_G$ are macroscopic observables. Physics: phase transitions happen at roots.

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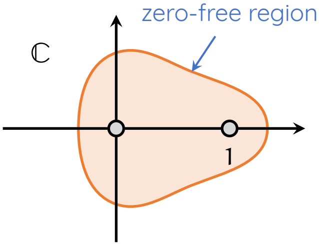

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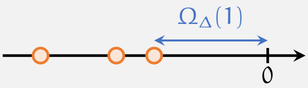

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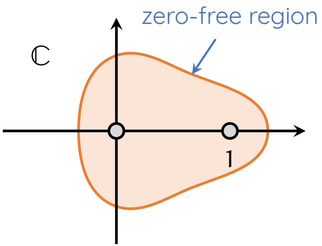

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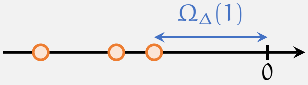

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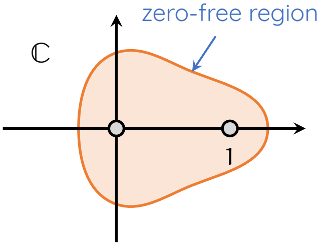

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- Approximate $p(1)$ using low-order derivatives of p at 0. 🙂

- Idea: truncate Taylor of $\log p$:

$$\log p(z) = a_0 + a_1 z + \dots$$

where $k! \cdot a_k = \frac{d^k}{dz^k} \log p(0)$ is a function of $p^{(0)}(0), \dots, p^{(k)}(0)$.

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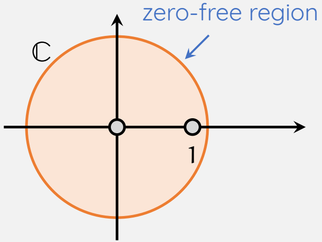

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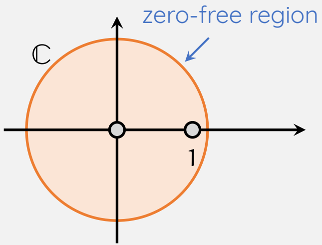

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For $1+\epsilon$ approx of $p(1)$, set

$$k = O\left(\frac{\log(\deg(p)/\epsilon)}{\delta}\right)$$

Proof:

- Since p is polynomial we can write
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- With [Patel-Regts], runtime is $O(1)^{O(\log(n/\epsilon))} = \text{poly}(n, 1/\epsilon)$. 🙂

How to extend beyond disks?

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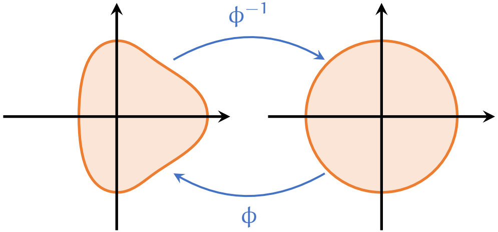
There is a biholomorphic map between any two simply connected regions in $\mathbb{C}$:



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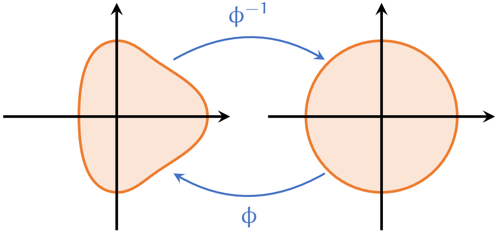

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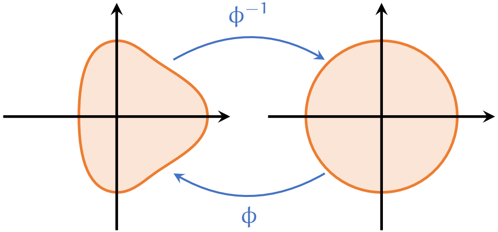

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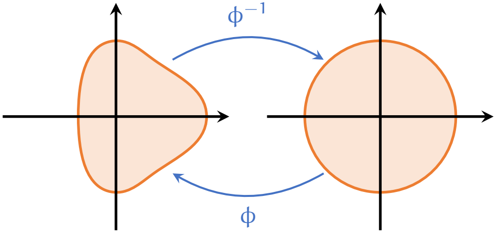

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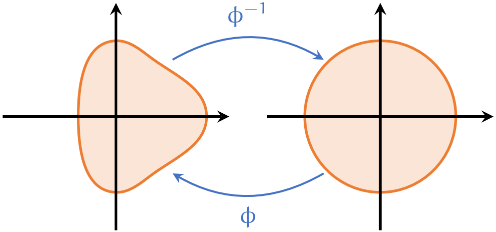

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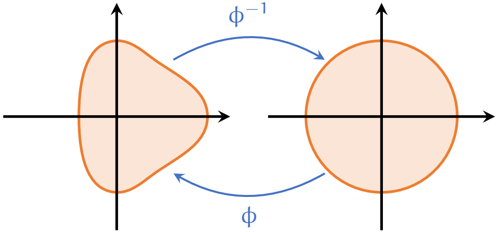

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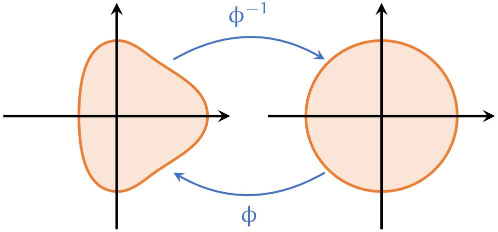

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Example: matching polynomial

- Region is $\mathbb{C} - \mathbb{R}_{\leqslant -r}$ for some r.
- Start with Möbius map

$$\phi(z) = (az + b)/(cz + d)$$

- Set a, b, c, d to ensure $\phi(0) = 0, \phi(1) = 1, \phi(\text{disk}) \cap \mathbb{R}_{\leqslant -r/2} = \emptyset$.
- Exercise: Taylor approx ϕ and compose with linear fn to get ψ.

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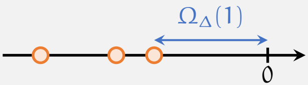
Theorem [Heilmann-Lieb'72]

Zeros/roots of p_G are real. In fact they lie in $(-\infty, -\Omega_\Delta(1)]$.



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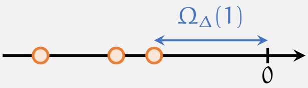

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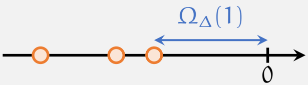

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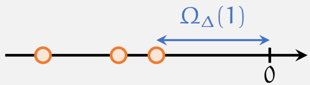

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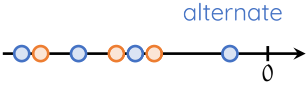

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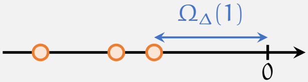

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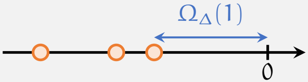

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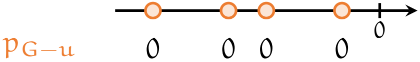

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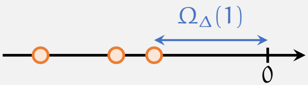

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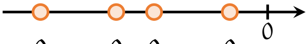

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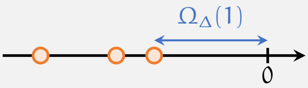

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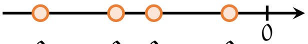

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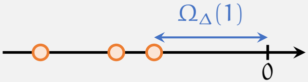

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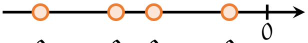

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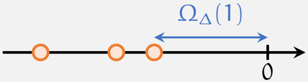

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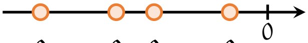

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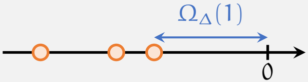

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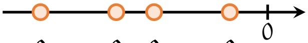

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- Next prove for $z \in (-\frac{1}{4\Delta}, 0]$:

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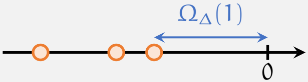


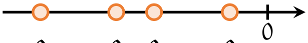
Proof:

- Idea: induction. Let u be vertex:

$$p_G = p_{G-u} + z\sum_{v\sim u} p_{G-u-v}$$

- Inductive claim: roots of p_G and p_{G-u} are real and interlace:

- Apply induction to p_{G-u} and p_{G-u-v}. Signs at roots of p_{G-u}:



p_{G-u}	0	0	0	0
p_{G-u-v_1}	$-$	$+$	$-$	$+$
p_{G-u-v_2}	$-$	$+$	$-$	$+$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_G	$+$	$-$	$+$	$-$

- By sign alts, we get interlacing of roots for p_G and p_{G-u}. ☺
- Next prove for $z \in (-\frac{1}{4\Delta}, 0]$:

$$2p_G(z) > p_{G-u}(z) > 0.$$

- Induction step:

$$p_G(z) \geqslant (1+2\Delta z)p_{G-u} \geqslant \tfrac{1}{2}p_{G-u}(z)$$

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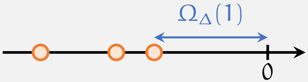

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alternate

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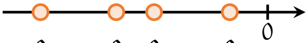

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- No roots $\geqslant -1/4\Delta$ ☺