CS 263: Counting and Sampling

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slides for

Log-Concave Polynomials

Review

Useful settings for local-to-global:

Spectral [A-Liu-OveisGharan]

$$\chi^2(\nu D_{k\to 1} \parallel \mu D_{k\to 1}) \leqslant \frac{C\,\chi^2(\nu\|\mu)}{k}$$

and similar inequalities for links is called C spectral independence.

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Special case of trickle-down

If $k \geqslant 3$ and links $\mu_{\{i\}}$ are 1-SI, then μ is either 1-SI, or $\lambda_2(U_{1\to k} D_{k\to 1}) = 1$.

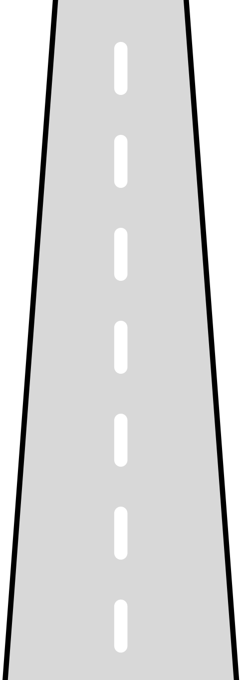
disconnected

Trickle Down

- Simplicial localization
- Covariance evolution
- Linear tilts

Log-Concave Polynomials

- Generating polynomials
- Exponential tilts
- Entropic independence



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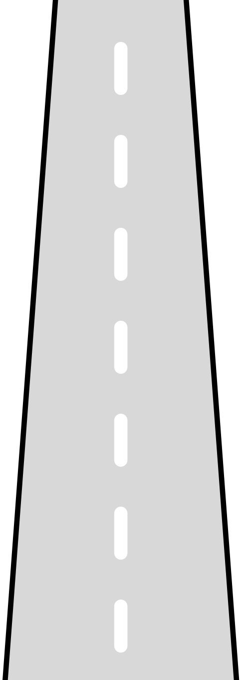

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Let $S \sim \mu$, and let $e_1, \dots, e_k$ be a u.r. permutation of S. Define ν_i as conditional of μ on $\{e_1, \dots, e_i\}$. Then

$$\mu = \nu_0 \to \nu_1 \to \nu_2 \to \cdots \to \nu_k$$

is called simplicial localization. ← we used this for local-to-global

Covariance evolution

- First step: $\mu \to \nu$. We know $\mathsf{mean}(\mu) = \mathbb{E}[\mathsf{mean}(\nu)]$. What about $\mathsf{cov}(\mu)$?

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- So $\mathbb{E}_i[\mathrm{mean}(\nu)\,\mathrm{mean}(\nu)^\intercal]$ is $\mathrm{mean}(\mu)\,\mathrm{mean}(\mu)^\intercal + \mathrm{cov}(\mu)\,\mathrm{cov}(w)\,\mathrm{cov}(\mu)$.

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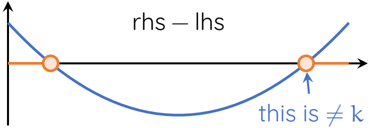

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- Expanding this and using $X = \Pi^{-1/2}\,\mathrm{cov}(\mu)\Pi^{-1/2}$ gives refined ineq:

$$X \preceq C \cdot \left(\frac{k-2}{k-1} \cdot I + \frac{2}{k(k-1)} \cdot X - \frac{1}{k(k-1)} \cdot X^2\right) + \frac{1}{k}X^2$$

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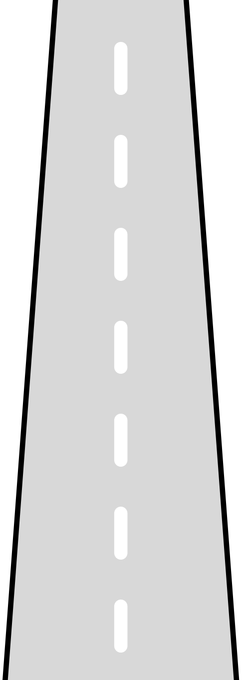
- Simplicial localization
- Covariance evolution
- Linear tilts

Log-Concave Polynomials

- Generating polynomials
- Exponential tilts
- Entropic independence



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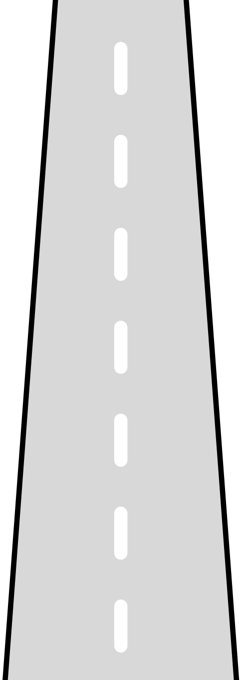

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$$\mu : \binom{[n]}{k} \to \mathbb{R}_{\geqslant 0}$$

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$$(\nabla^2 g_\mu(\mathbb{1}))_{ij} \propto \mathbb{1}[i \neq j] \cdot \mathbb{P}_{S \sim \mu}[i, j \in S]$$

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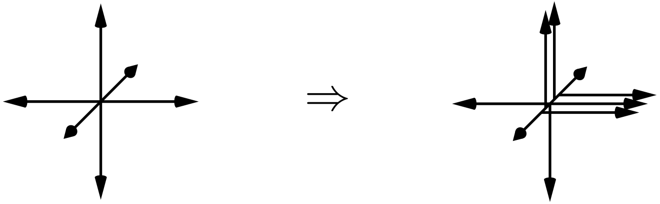
- If $\deg = 2$ derivatives h are log-concave at $\mathbb{1}$, and links are connected, then g_μ is log-concave at $\mathbb{1}$. [Oppenheim, A-Liu-OveisGharan-Vinzant, Brändén-Huh]

in fact everywhere

Exponential tilt/external field



$$\mu(x) \mapsto \nu(x) \propto \underbrace{\exp(\langle w, x\rangle)}_{\text{exponential tilt}} \mu(x)$$

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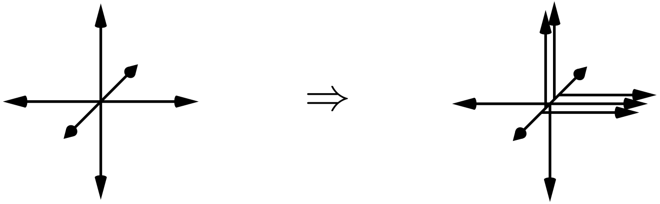

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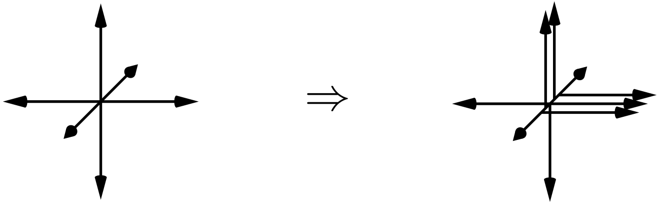

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- If for $\deg = 2$ derivatives h of g_μ, we have $\lambda_2(\nabla^2 h) \preceq 0$, we do for g_ν too:
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- Corollary: g_ν is log-concave at $\mathbb{1}$. This means g_μ is log-concave on $\mathbb{R}^n_{>0}$.

Theorem [Oppenheim, A-Liu-OveisGharan-Vinzant, Brändén-Huh]

g_μ is log-concave on $\mathbb{R}^n_{>0}$ iff all of its $\deg = 2$ derivatives are log-concave and all derivatives are connected.

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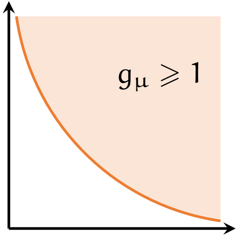

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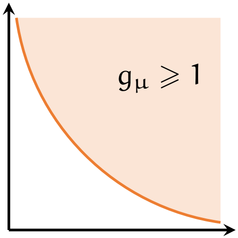

- So far we know matroids are 1-SI, and thus $t_{\text{mix}} = O(k^2 \log n)$. Can we improve?

Informal theorem [A-Jain-Koehler-Pham-Vuong'21]

"Improved mixing time" assuming μ is spectrally independent under all external fields.

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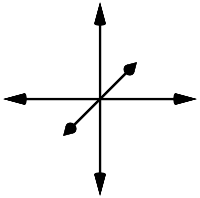
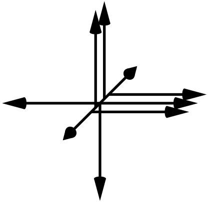


$\Rightarrow$



Goal

Bound ρ_{entropy} by ρ_{variance}.

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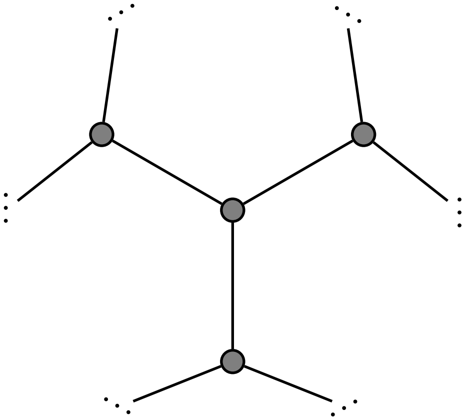
- Impossible with no assumption ☹

For constant degree expanders

$$\rho_{\text{variance}} = \Omega(1),$$

but mixing time is

$$\simeq \log(|\text{state space}|)\,.$$



Entropic Independence

For all distributions ν,

$$\mathcal{D}_{\mathsf{KL}}(\nu D_{k\to 1} \parallel \mu D_{k\to 1}) \leqslant \frac{C}{k} \cdot \mathcal{D}_{\mathsf{KL}}(\nu \parallel \mu).$$

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- The greatest aspect of entropic independence: no need to consider all ν. Enough to look at

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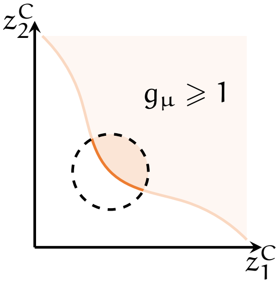
for $w \in \mathbb{R}^n$. This is an external field applied to μ.
- If $q = (q_1, \dots, q_n)$ is some distribution on $[n] = \binom{[n]}{1}$, then
$$\inf\{\mathcal{D}_{\mathrm{KL}}(\nu \parallel \mu) \mid \nu D_{k\to 1} = q\} = -\log\left(\inf_{z_1,\dots,z_n>0} \frac{\sum_S \mu(S) \prod_{i\in S} z_i}{z_1^{kq_1} \cdots z_n^{kq_n}}\right).$$

Polynomial view of HDX

$$g_\mu(z_1,\ldots,z_n) := \sum_S \mu(S) \prod_{i\in S} z_i$$

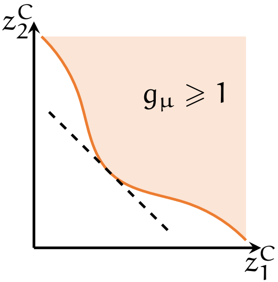
Spectral Independence

- Local: level sets of $g_\mu(\sqrt[c]{z_1},\ldots,\sqrt[c]{z_n})$ locally convex at $\mathbb{1}$.



Entropic Independence

- Global: level sets of $g_\mu(\sqrt[c]{z_1},\ldots,\sqrt[c]{z_n})$ bounded by tangent at $\mathbb{1}$.



Theorem [A-Jain-Koehler-Pham-Vuong'21]

C-spectral independence for external fields $\implies$ C-entropic independence.