CS 263: Counting and Sampling

Nima Anari

Stanford University

slides for

Spectral Independence

Review

- Dist μ on $\binom{U}{k}$ ← simplicial complex

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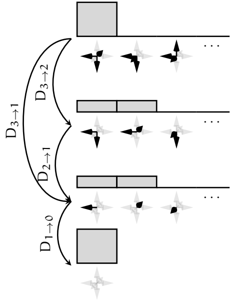
- Down kernels



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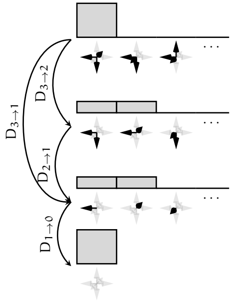

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- Up kernels:

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  $D_{k\to\ell}U_{\ell\to k}$

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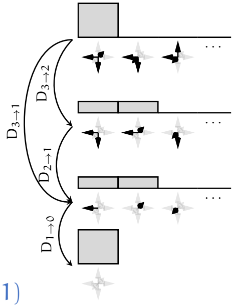

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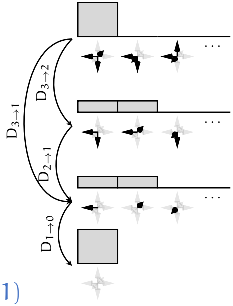

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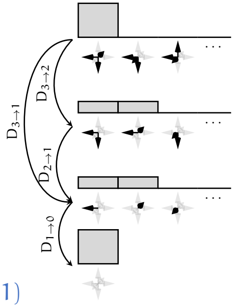

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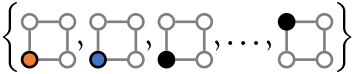
$U =$ 

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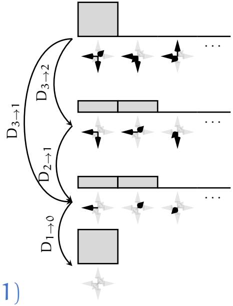

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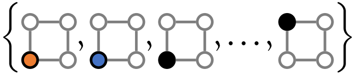
  $U =$ 
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  $$\mathrm{dist}_{S \sim \mu}(S \mid T \subseteq S)$$

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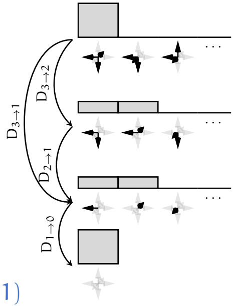

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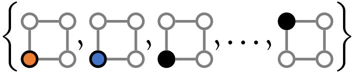
  $U =$ 

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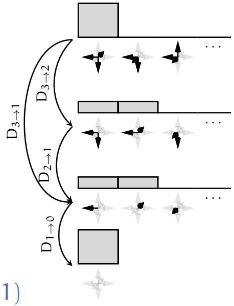

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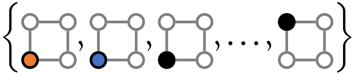
  $U =$ 
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Local to global

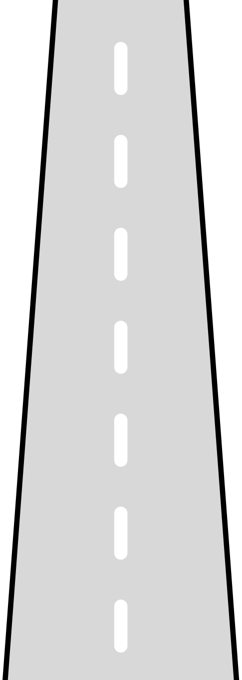
If μ_T has local contraction $1-\rho_{|T|}$, then we get global contraction $1-\rho$ where

↑ for $D_{k\to\ell}$

$$\rho = \rho_0 \cdots \rho_{\ell-1}$$

Spectral HDX Analysis

- Spectral independence
- Entropic independence
- Trickle down
- Matroids



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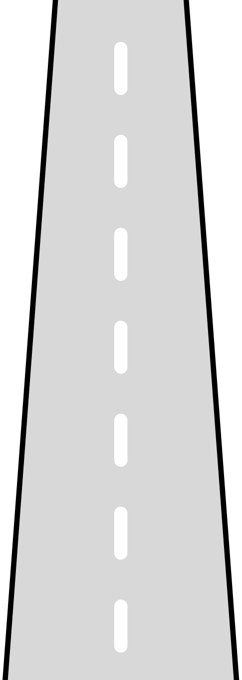

Spectral and entropic independence

Useful settings for local-to-global:

Spectral [A-Liu-OveisGharan]

$$\chi^2(\nu D_{k\to 1} \parallel \mu D_{k\to 1}) \leqslant \frac{C\,\chi^2(\nu \| \mu)}{k}$$

and similar inequalities for links is called C spectral independence.

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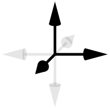
- Will show matroids satisfy $C = 1$.



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The matrix Ψ with entries $\mathbb{P}[j \mid i] - \mathbb{P}[j]$.

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$$\lambda_{\max}(\Psi) \leqslant C$$

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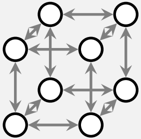

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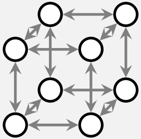

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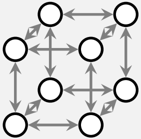

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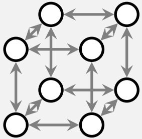

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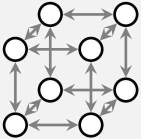

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$$(D\Psi D^{-1})_{ij} = \frac{\mathbb{P}[i,j]-\mathbb{P}[i]\,\mathbb{P}[j]}{\sqrt{\mathbb{P}[i]\,\mathbb{P}[j]}}$$

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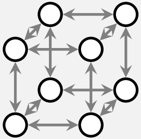

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- Spectral independence is equiv to

$$D\Psi D^{-1} \preceq C \cdot I$$

which is the same as

$$D^2\Psi \preceq C \cdot D^2.$$

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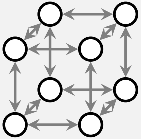

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- So $D_{n\to\ell}$ contracts χ^2 by ℓ/n.

- Ψ also has a symmetric form $D\Psi D^{-1}$. With D diagonal and $D_{ii} = \sqrt{\mathbb{P}[i]}$, we get

$$(D\Psi D^{-1})_{ij} = \frac{\mathbb{P}[i,j]-\mathbb{P}[i]\,\mathbb{P}[j]}{\sqrt{\mathbb{P}[i]\,\mathbb{P}[j]}}$$

- Spectral independence is equiv to

$$D\Psi D^{-1} \preceq C \cdot I$$

which is the same as

$$D^2\Psi \preceq C \cdot D^2.$$

- If we embed $\binom{U}{k} \hookrightarrow \{0,1\}^U_{\|\cdot\|_1=k}$, by sending S to $\mathbb{1}_S$, the inequality becomes

$$\mathrm{cov}(\mu) \preceq C \cdot \mathrm{diag}(\mathrm{mean}(\mu))$$

How to establish spectral independence?

- Transport contraction
- Transport stability
- Correlation decay
- Geometry of polynomials
- Trickle down ← today
- ...
- Universality: any linear bound on t_{rel} for down-up

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If $k \geqslant 3$ and links $\mu_{\{i\}}$ are 1-SI, then μ is either 1-SI, or $\lambda_2(U_{1\to k}D_{k\to 1}) = 1$.

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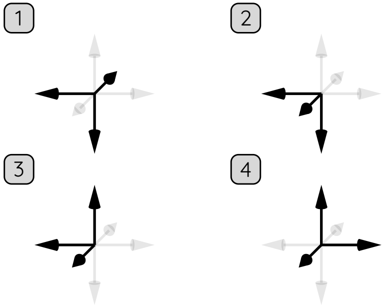

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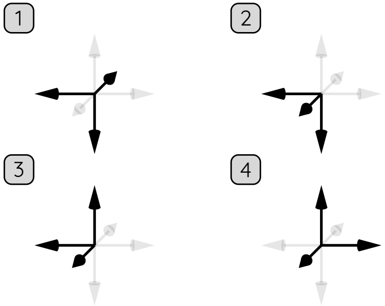
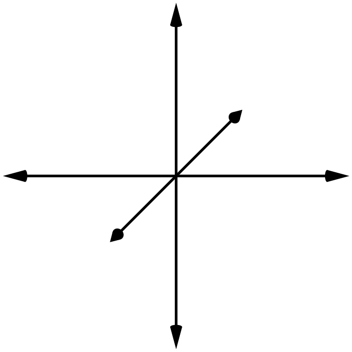


- Can reach any T from any S by exchanges.

Rank 2 matroids



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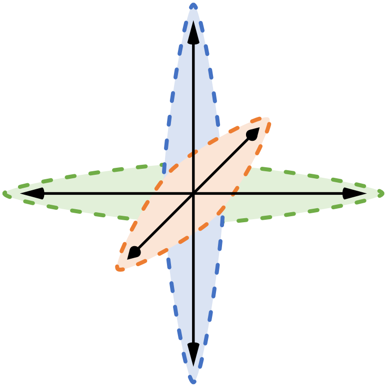

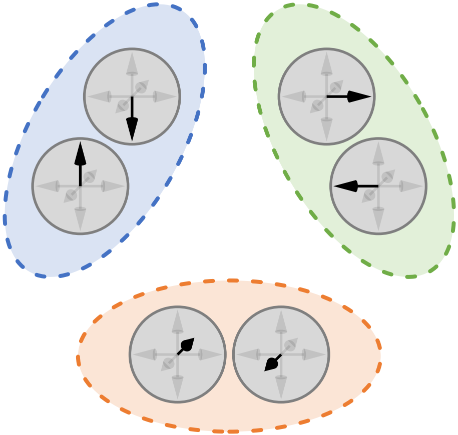

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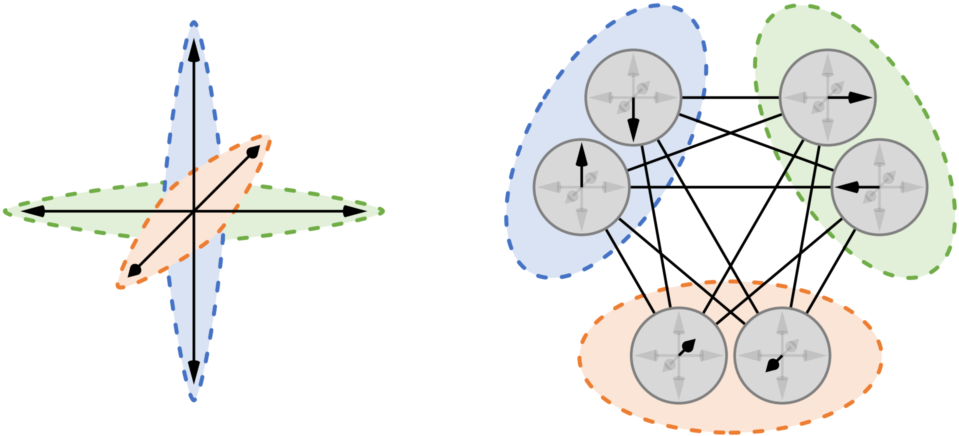

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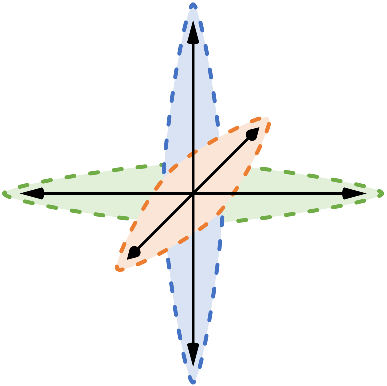

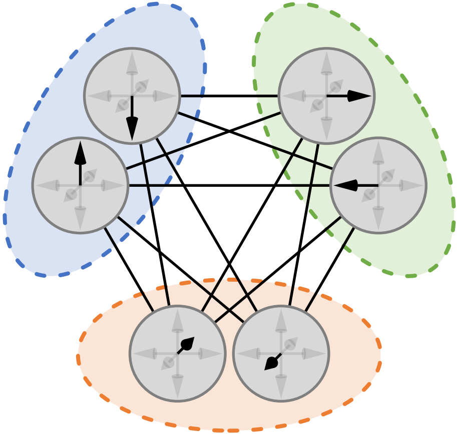

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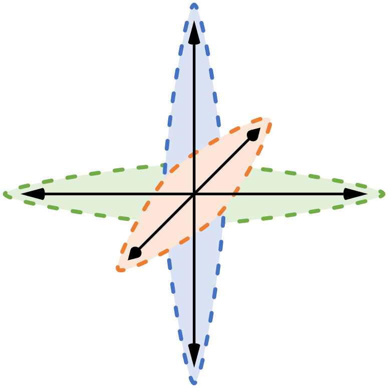

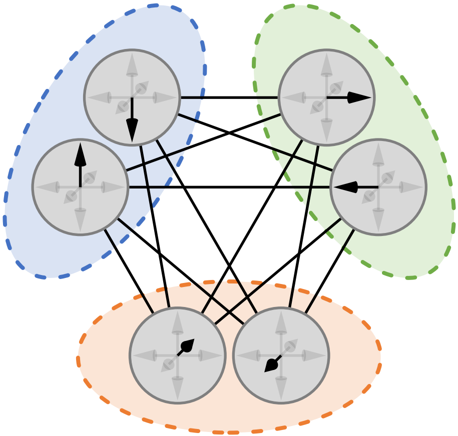

- Distribution $\mu \equiv$ uniformly random edge of complete multipartite graph.
- The walk $U_{1\to 2}D_{2\to 1}$: lazy random walk on complete multipartite graph.

- These graphs have adj matrix:

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