CS 263: Counting and Sampling

Nima Anari

Stanford University

slides for

Bipartite Perfect Matchings

Review

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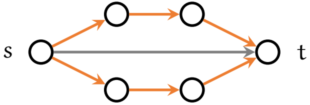

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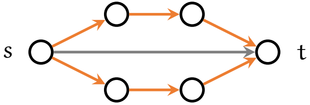

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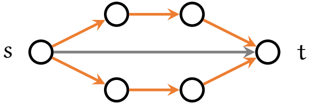

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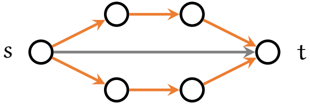

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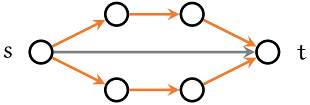

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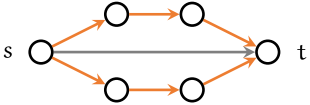

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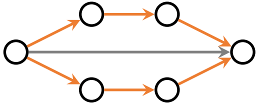

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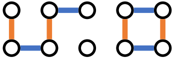
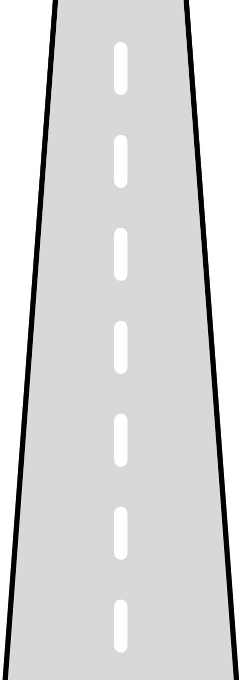
- Matching walks mix in $\mathsf{poly}(n)$.



Perfect Matchings

- Monomer-dimer systems
- Log-concave sequences
- Bipartite graphs



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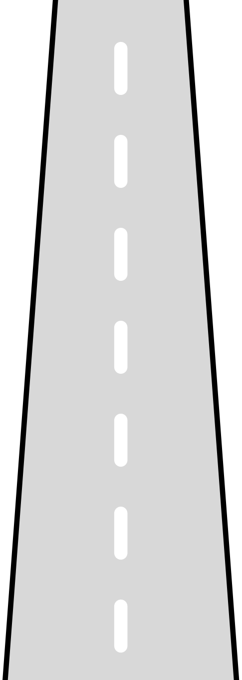

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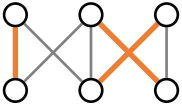

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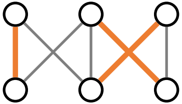

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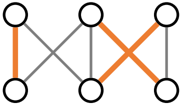

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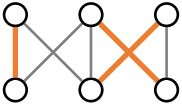

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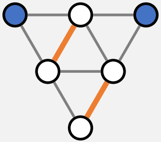

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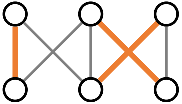

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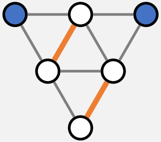

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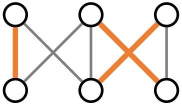

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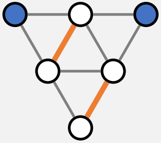

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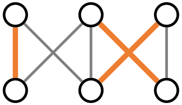

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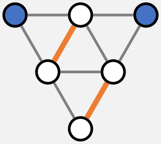

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- Sampling/counting possible in $\mathbf{poly}(n, \log \lambda_{\max})$ time on bipartite graphs [Jerrum-Sinclair-Vigoda]. ☺

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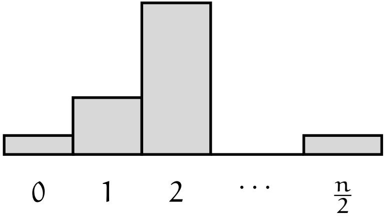

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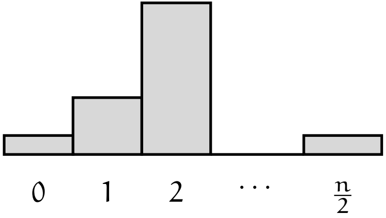

- If $\lambda^k \cdot \#(k\text{-matchings})$ maximized for $k = \frac{n}{2}$, use rejection sampling.

Fact: log-concavity of matchings

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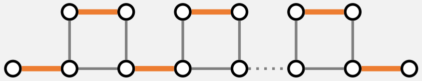

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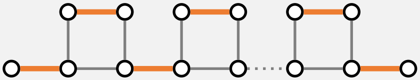

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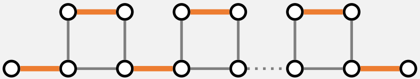

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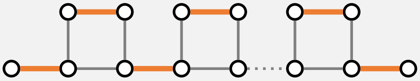

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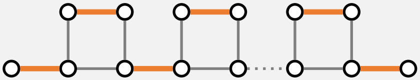

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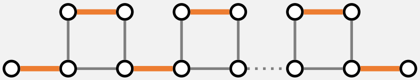

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- [Jerrum-Sinclair-Vigoda'04] showed this works on bipartite graphs.

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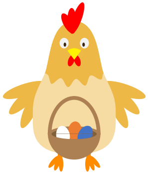

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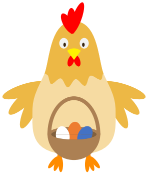

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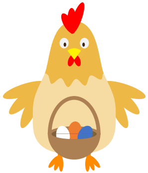

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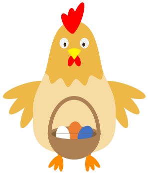

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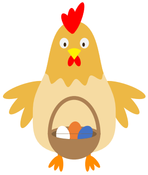

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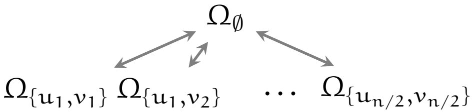

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- Use the same encoding as before:

$$\mathrm{enc}(s,t) = s \oplus t \oplus x - \text{couple edges}$$

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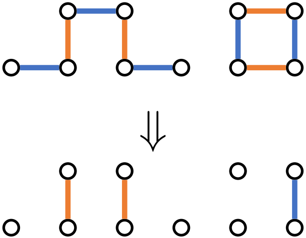

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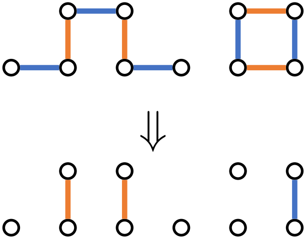

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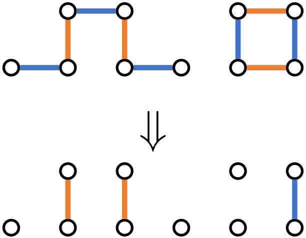

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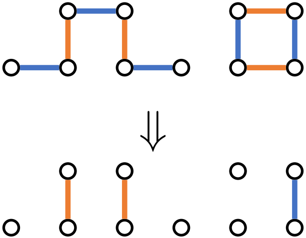

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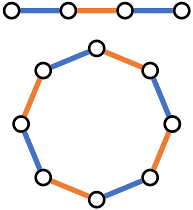
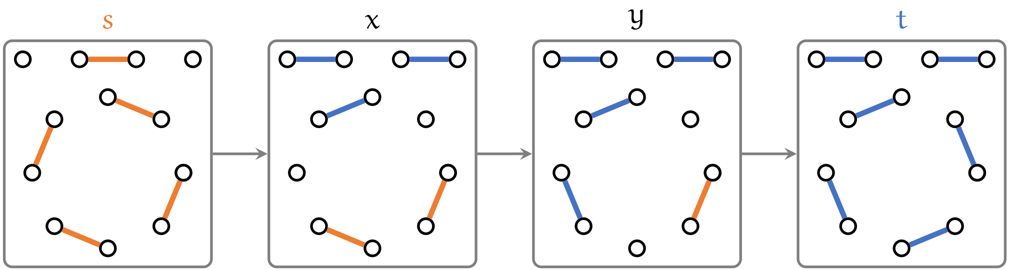
$$\sum_{|S|\leqslant 4} \mu(\Omega_S) \leqslant \text{poly}(n) \cdot \sum_{|S|\leqslant 2} \mu(\Omega_S)$$

- We just need to show $\mu(s)\mu(t) \leqslant$

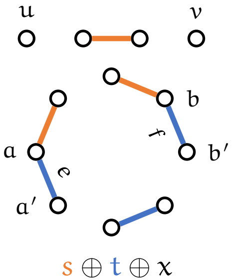
$$\text{poly}(n) \cdot \min\{\mu(x), \mu(y)\} \cdot \mu(\text{enc})$$

- It's a bit of case analysis, but hardest case is in the middle of unraveling a cycle:

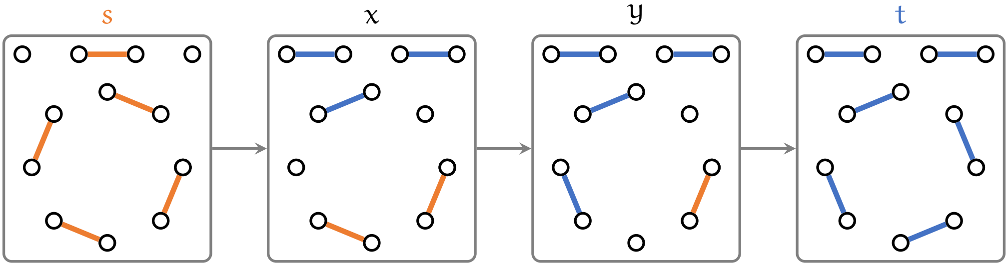




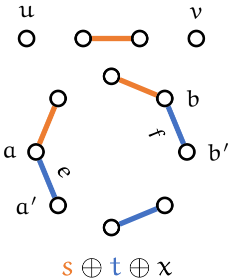
s
x
y
t

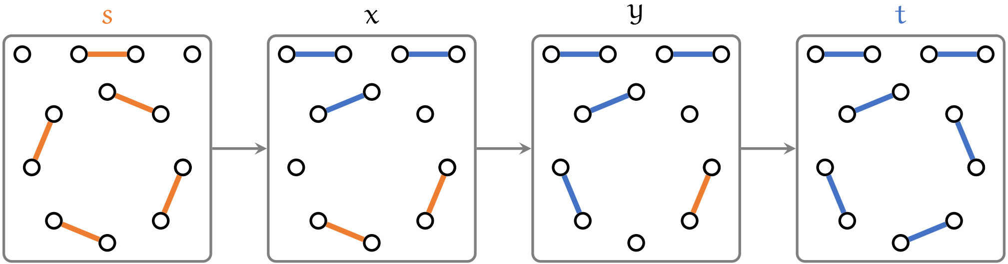


u
v
b
f
a
e
b′
a′
s ⊕ t ⊕ x

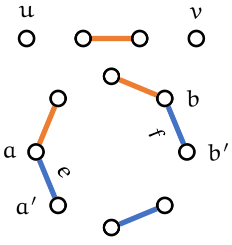


$\mathsf{enc} = s \oplus t \oplus x - e - f$





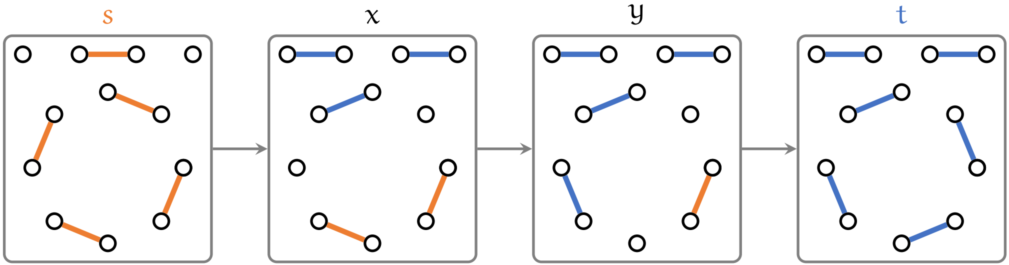
$$\mathsf{enc} = s \oplus t \oplus x - e - f$$



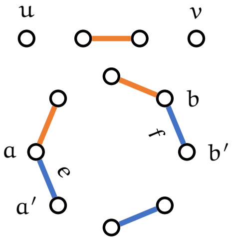
$s \oplus t \oplus x$

- Note that $\lambda^s \lambda^t = \lambda_e \lambda_f \lambda^x \lambda^{\mathsf{enc}}$. Let e's endpoints be a, a' and f's endpoints be b, b'. Prove: ⟵ via injective map

$$\lambda(\underset{\substack{\uparrow\\ t}}{\Omega_{\emptyset}})\lambda(\underset{\substack{\uparrow\\ s}}{\Omega_{\{u,v\}}}) \geqslant \frac{1}{\mathsf{poly}(n)} \cdot \lambda_e \lambda_f \lambda(\underset{\substack{\uparrow\\ x}}{\Omega_{\{a,b\}}})\lambda(\underset{\substack{\uparrow\\ \mathsf{enc}}}{\Omega_{\{u,v,a',b'\}}})$$



$$\mathsf{enc} = s \oplus t \oplus x - e - f$$



$s \oplus t \oplus x$

- Note that $\lambda^s \lambda^t = \lambda_e \lambda_f \lambda^x \lambda^{\mathsf{enc}}$. Let e's endpoints be a, a' and f's endpoints be b, b'. Prove: $\longleftarrow$ via injective map

$$\lambda(\underset{\overset{\uparrow}{t}}{\Omega_\emptyset})\lambda(\underset{\overset{\uparrow}{s}}{\Omega_{\{u,v\}}}) \geqslant \frac{1}{\mathsf{poly}(n)} \cdot \lambda_e \lambda_f \lambda(\underset{\overset{\uparrow}{x}}{\Omega_{\{a,b\}}})\lambda(\underset{\overset{\uparrow}{\mathsf{enc}}}{\Omega_{\{u,v,a',b'\}}})$$

- Thus $\mu(s)\mu(t) \leqslant \mathsf{poly}(n) \cdot \mu(x)\mu(\mathsf{enc})$. Similar ineqs yield $\mu(s)\mu(t) \leqslant \mathsf{poly}(n) \cdot \mu(y)\mu(\mathsf{enc})$. So cong $\leqslant \mathsf{poly}(n)$.