CS 263: Counting and Sampling

Nima Anari

Stanford University

slides for

Introduction

Logistics

- Course staff:

Nima Anari
(Instructor)

Victor Lecomte
(Course Assistant)

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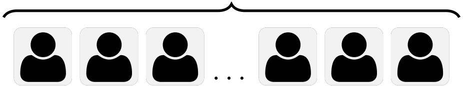

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- Office hours: Starting next week

What is “Counting and Sampling”?

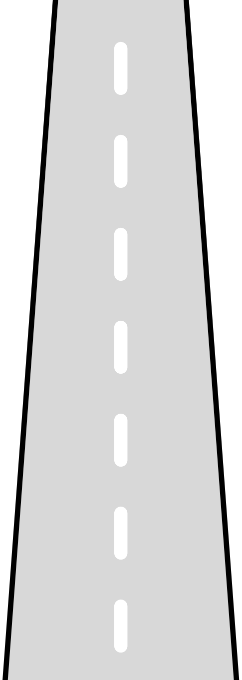
Bit of Complexity Theory

- The class #P
- Parsimonious reductions

Approximation

- Counting: FPTAS/FPRAS
- Sampling: FPAUS
- Equivalence

First Algorithm: DNFs



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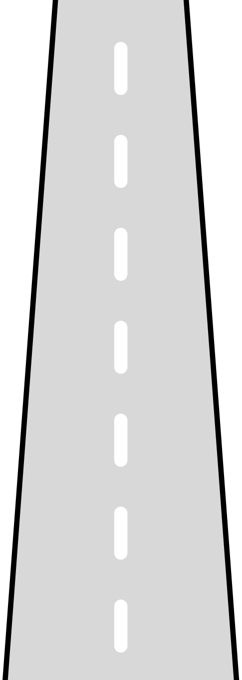

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Distribution μ on large Ω

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- Counting: efficiently computing $\mathbb{P}_{\mu}[\text{event}]$ for events of interest.

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Example: #SAT

$$\phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4) \wedge \cdots$$

- Ω is $\{0,1\}^n$.
- μ is uniform over satisfying assignments.

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- In fact, numerator is #sat assignments to

$$\phi' = \phi \wedge x_1.$$

This is called "self-reducibility".

will come back to this later

Formalism

w.r.t. an easy background measure on Ω, usually counting/uniform on finite Ω

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Standard assumption: μ is easy to compute for any desired point $\omega \in \Omega$.

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- $\Omega = \{0,1\}^n$ ← assignments
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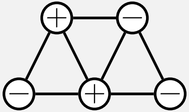
Example: spin systems



graph $G = (V, E)$

- $\Omega = \{+,-\}^V$ ⟵ could be larger
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local interaction

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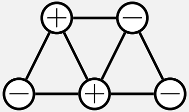

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Example: generative AI models

- $\Omega = \{$, , , $\ldots\} \longleftarrow$ images
- $\Omega = \{\text{good job, slay, sus}, \ldots\} \longleftarrow$ text

We don't know μ. We learn something about it from data. What to learn is often guided by a sampling algorithm.

- Score-based models: $\nabla \log \mu$

nearby points

$x \qquad x + \Delta x$

$$\frac{\mu(x + \Delta x)}{\mu(x)} \simeq \exp(\nabla \log \mu \cdot \Delta x).$$

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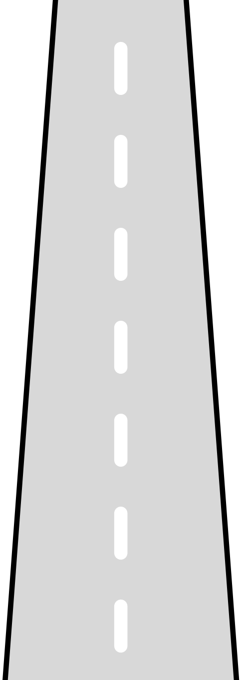

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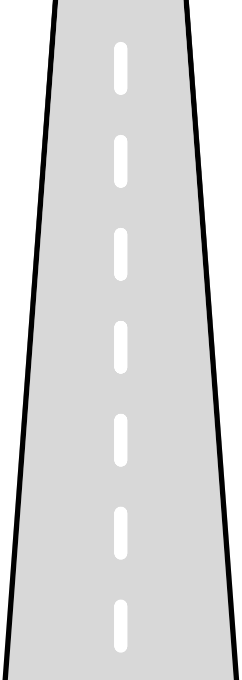
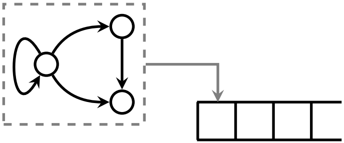


Poly-time nondet. Turing machine M



$M : (x, y) \mapsto \{\text{Accept}, \text{Reject}\}$

input

witness/nondet. choices

Example: SAT

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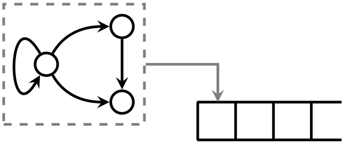

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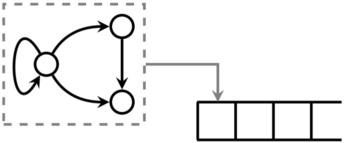

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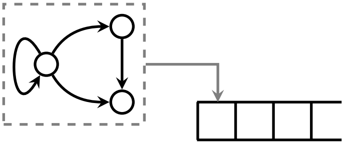

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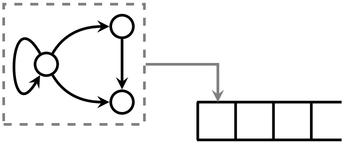

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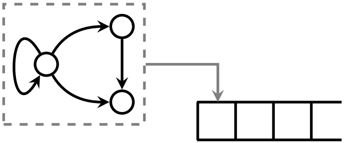

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#3-Colorings	#Matchings
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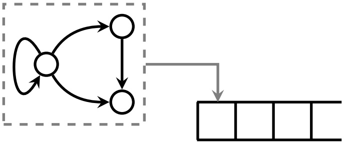

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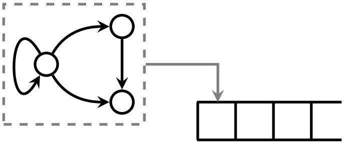

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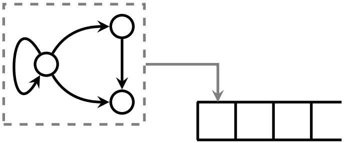

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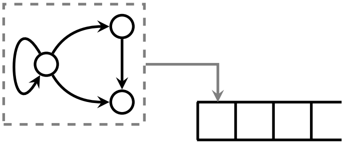

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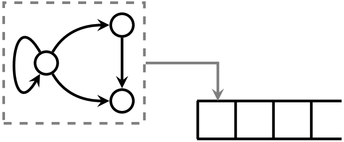

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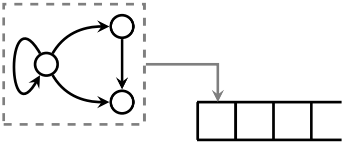

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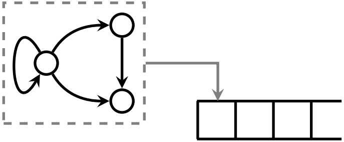

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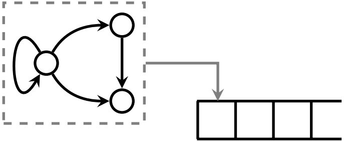

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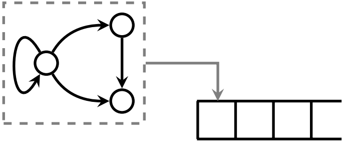

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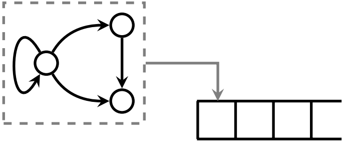

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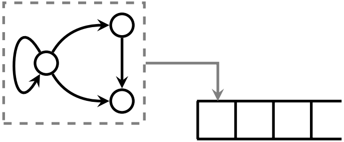

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- Open problem: Do all NP-complete problems have a #P-complete variant?

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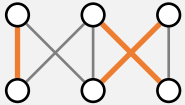

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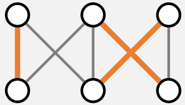

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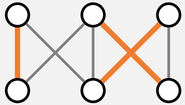

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- Reductions are not parsimonious.
- Observation: efficient counting known for only a handful of gems: spanning trees, planar perf. matchings, Eulerian circuits, ...

will come back to them

All hope is lost?

What is “Counting and Sampling”?

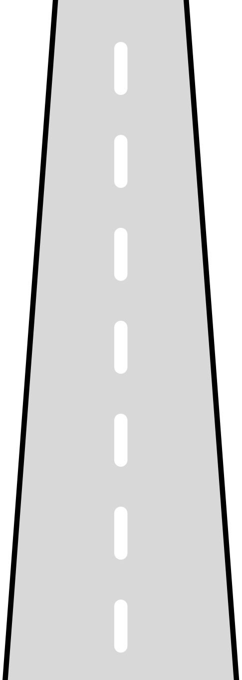
Bit of Complexity Theory

- The class #P
- Parsimonious reductions

Approximation

- Counting: FPTAS/FPRAS
- Sampling: FPAUS
- Equivalence

First Algorithm: DNFs



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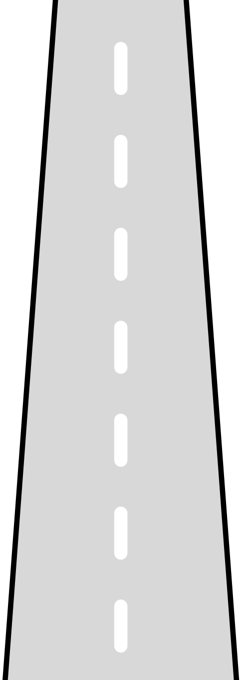

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Example: #SAT

Suppose A is $f(n)$-approx. alg. Give
$$\phi^{(1)} \wedge \phi^{(2)} \wedge \cdots \wedge \phi^{(t)}$$
with $\phi^{(i)}$ being disjoint copies of ϕ.
$$\sqrt[t]{\text{output}} \approx \#\mathsf{SAT}(\phi).$$
Approx. ratio is $\sqrt[t]{f(nt)}$. Even for $f(n) = 2^{n^{0.99}}$, enough to set $t = \mathsf{poly}(n, 1/\epsilon)$ to get $\sqrt[t]{f(nt)} \leqslant 1+\epsilon$.

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- For any "tensorizable" problem, nothing between $1+\epsilon$ and exponential.
- For "self-reducible" probs, $\mathsf{poly}(n)$-approx gives an FPRAS. [Jerrum-Sinclair].
  will see later in the course

Approximate sampling

- Notion of approximation: For dists ν, μ on Ω we use total variation:

$$\begin{aligned}
&d_{TV}(\nu, \mu) \\
&= \max\{\mathbb{P}_\nu[E] - \mathbb{P}_\mu[E] \mid E \subseteq \Omega\} \\
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Theorem [Jerrum-Valiant-Vazirani]

For "self-reducible" problems:

approx counting $\equiv$ approx sampling

$$
\begin{tikzcd}
& \text{(FPRAS)} \\
\text{Exact Counting} \arrow[r] \arrow[d] & \text{Approx Counting} \arrow[dl, "\text{if FPTAS}"'] \arrow[d, leftrightarrow] \\
\text{Exact Sampling} \arrow[r] & \text{Approx Sampling} \\
& \text{(FPAUS)}
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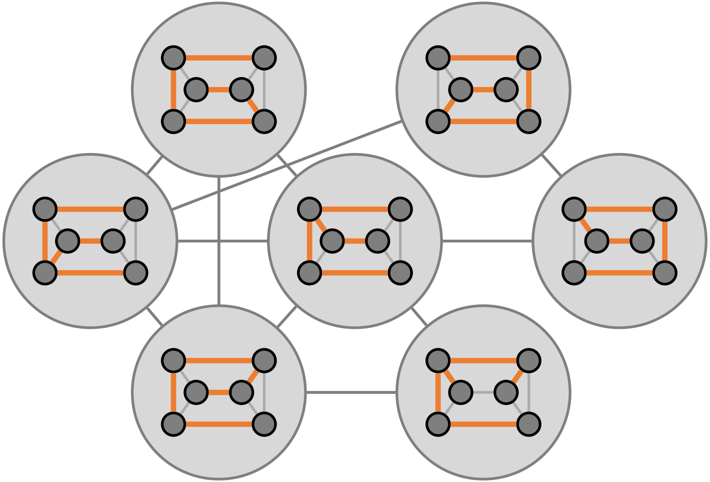
arrows are poly-time reductions

Counting via Markov chains

- Basis of Markov Chain Monte Carlo: Approx Sampler → Approx Counter.



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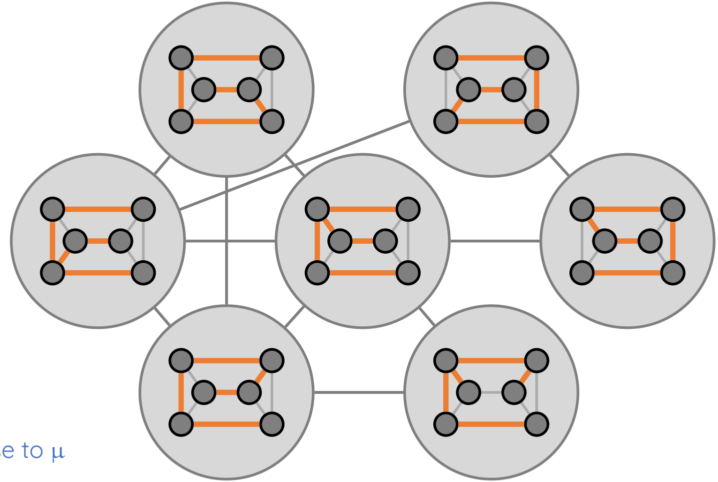
- Basis of Markov Chain Monte Carlo: Approx Sampler → Approx Counter.
- A good portion of this course will be on sampling via Markov chains.

$$x_0 \to x_1 \to \cdots \to x_t$$

hope this is close to μ



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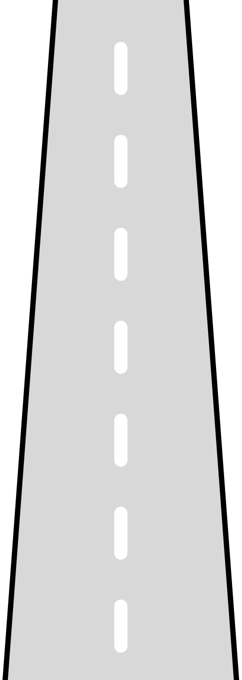

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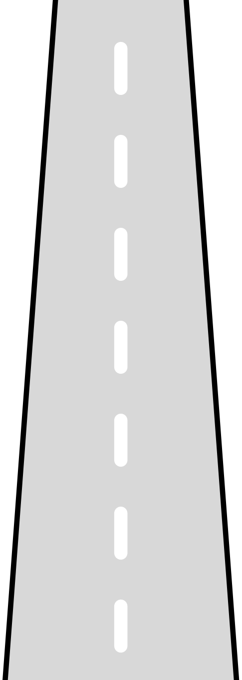

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- For $\phi = (x_1 \wedge x_2 \wedge \cdots \wedge x_n)$ it takes 2^n tries on average. ☹️

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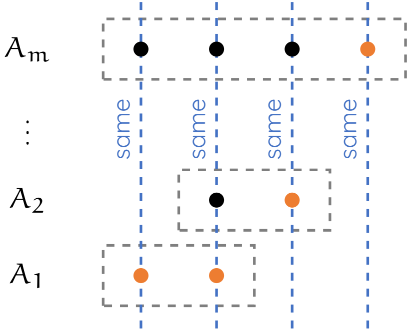

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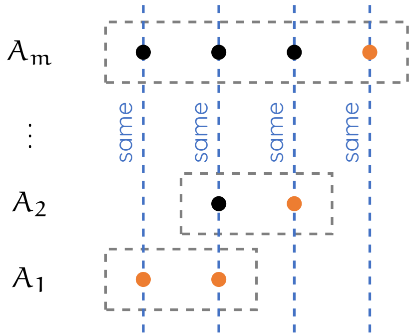

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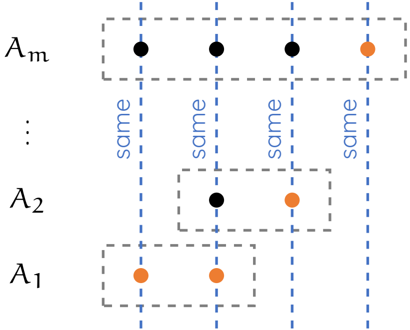

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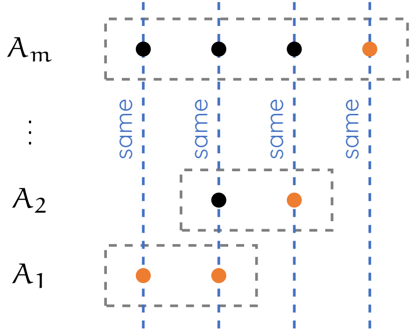
  if x *is sampled from* A_i *and* $x \notin A_j$ *for all* $j < i$ **then**
    accept and return x



- Chance of acceptance $\geqslant 1/m$.
- On average $\simeq m$ iterations suffice.
- Next lecture: turning this into approx counting.