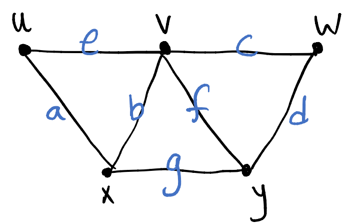
Review

- Equiv. of counting & sampling: [self-reducible probs]

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Exact Counting} \arrow[r] \arrow[d] & \text{Approx Counting (FPRAS)} \arrow[dl, dashed, "\text{if FPTAS}"'] \arrow[d, leftrightarrow, "\text{[Jerrum-Valiant-Vazirani]}"] \\
\text{Exact Sampling} \arrow[r] & \text{Approx Sampling (FPAUS)}
\end{tikzcd}
$$

- Matrix-tree thm.
  * $(n-1)\times(n-1)$ submatrix has
    $\begin{cases} 0 \text{ det} & \text{if cycle} \\ \pm 1 \text{ det} & \text{if tree} \end{cases}$
  * Use Cauchy-Binet
    $\det(AA^T) = \sum_{S} \det(A_S)^2$
    S: subset of columns



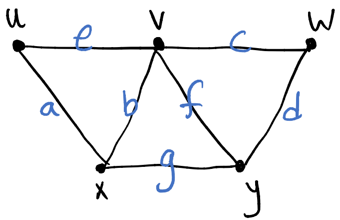
$$
\begin{array}{c|c}
 & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{array}{c} u \\ v \\ w \\ x \\ y \end{array} &
\begin{bmatrix}
+1 & 0 & 0 & 0 & +1 & 0 & 0 \\
0 & -1 & +1 & 0 & -1 & -1 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & +1 & +1
\end{bmatrix}
\end{array}
$$

vertex-edge adj matrix

Plan for Today

- Quick remark about matrix-tree thm
- Counting via determinants II:
  * Bipartite planar matchings
- Intro to Markov chains

Matrix-Tree Thm



$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{matrix} u \\ v \\ w \\ x \\ y \end{matrix} & \begin{bmatrix} +1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 & +1 \end{bmatrix} \end{array}$$

vertex-edge adj.

Laplacian Matrix:

$$L := AA^T = \begin{array}{c|c} & \begin{matrix} u & v & w & x & y \end{matrix} \\ \begin{matrix} u \\ v \\ w \\ x \\ y \end{matrix} & \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix} \end{array}$$

non-edge edge degree

Matrix-tree thm:

det($(n-1)\times(n-1)$ principal submatrix of L) = # spanning trees

- Exact counting ✓
- Exact sampling ✓ ← by easy arrows in equivalence between counting & sampling

HW: Generalize to directed!

Runtime: $O(n^\omega)$ ← matrix multiplication exponent ≤ 3

Sampling: - naively $O(m) \times$ counting
- smarter: $\tilde{O}(n^\omega)$ [Colbourn-Myrvold-Neufeld '96]

Best known:

- $\tilde{O}(n^2)$ approx counting [Durfee-Peebles-Peng-Rao '17]
- $\tilde{O}(m)$ approx sampling [A-Liu-OveisGharan-Vinzant-Vuong '20]

Open: Improve counting!

Open: Speedups for directed graphs.

Bipartite Perfect Matchings

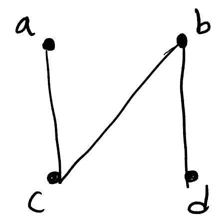
Valiant proved #P-complete.

Two options:

- Count approximately ← later in course
- Restrict graphs ← today

Permanent:

$$\mathrm{per}(A) := \sum_{\sigma:\ \text{permutation}} A_{1,\sigma(1)} \cdots A_{n,\sigma(n)}$$



$\mathrm{per}(\text{bipartite adj}) = \#\text{P.M.}$

$$\begin{array}{c|cc} & c & d \\ \hline a & 1 & 0 \\ b & 1 & 1 \end{array} \quad \text{i.e.} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Determinant:

$$\det(A) = \sum_{\sigma} \mathrm{sgn}(\sigma) A_{1,\sigma(1)} \cdots A_{n,\sigma(n)} \qquad \mathrm{per} = 1$$

Pólya's Scheme:

Replace 1s with ±1s to make all terms equal-signed.

Example. ($K_{2,2}$)

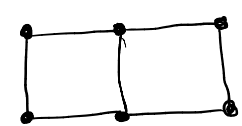
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}\right) = \mathrm{per}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

Non-Example. ($K_{3,3}$)

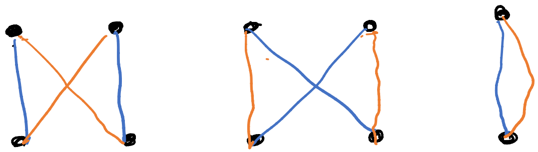
Impossible for all terms to have same sign. Exercise: show this!

Theorem. Possible for planar graphs.
[Kasteleyn '67, Fisher–Temperley '63]

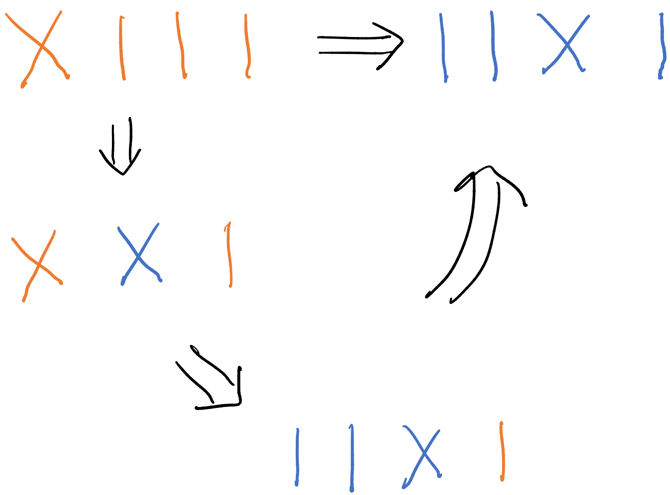


← e.g. 2D lattices

Moving Between PMs:



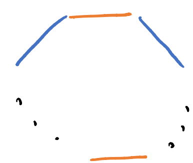
Orange and blue PMs. Make sure signs in det are the same.



- Can move from PM to PM by exchanging one cycle at a time.

Nice cycle: vertex-complement has perfect matching.

Desired: Signing where nice cycles do not change sgn(term).



← going from blue to orange does not change overall sign.

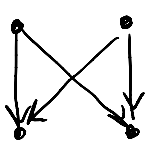
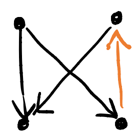
$$\sum \operatorname{sgn}(\sigma)\, A_{1,\sigma(1)} \cdots A_{n,\sigma(n)}$$

Note: $\operatorname{sgn}(\sigma)$ changes by $(-1)^{\text{len}/2+1}$

Pfaffian Orientation

- Direct edges from one side to other.
- Along any cycle #fwd oriented = len/2.
- Flip some directions ≡ Change +1 to −1.

Example.

 $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Claim. If nice cycles have odd many fwd. edges $\Longrightarrow$ sign does not change.

Proof. – $\frac{\text{len}}{2}$ odd: $\text{sgn}(\sigma)$ doesn't change

$$\prod_{e \in \text{cycle}} A_e = +1 \checkmark$$

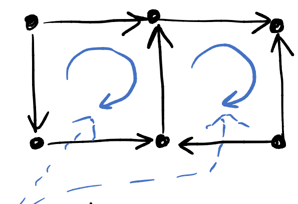
– $\frac{\text{len}}{2}$ even: $\text{sgn}(\sigma)$ changes

$$\prod_{e \in \text{cycle}} A_e = -1 \checkmark$$

Planar Graph

Goal: Find orientation with odd many clockwise in each nice cycle.

Example.

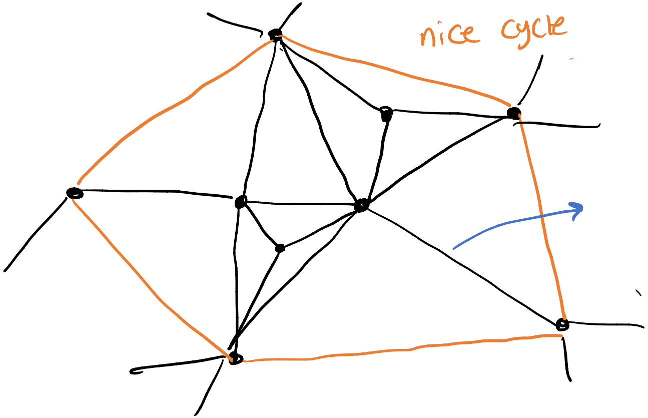


Lemma 1. Faces have odd many C.W.

$\Longrightarrow$ all nice cycles do.

Lemma 2. ∃ orientation where faces have odd-many C.W. This orientation can be found efficiently.

Faces $\Rightarrow$ All nice cycles



#(c.w around cycle) $\overset{2}{\equiv}$

$\sum_{f:\ \text{interior face}}$ #(c.w. around f) + #(interior edges) $\overset{2}{\equiv}$

#(interior faces) + #(interior edges) $\overset{2}{\equiv}$

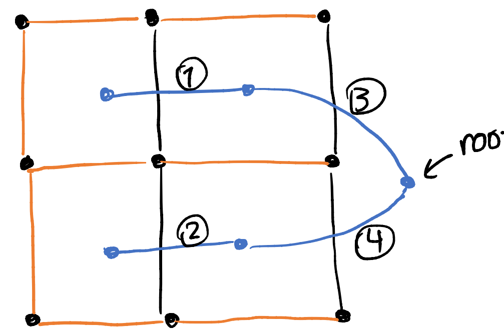
#(interior vertices) + 1 even by niceness!

Euler's formula: verts − edges + faces = 1 $\Rightarrow$

len + #(int verts) − len − #(int edges) + #(int faces) = 1

Good orientation for faces

- Assume connected w.l.o.g.
- Choose spanning tree.
- Gives dual spanning tree.



- Orient spanning tree arbitrarily.
- Orient duals of dual spanning tree one-by-one, peeling one leaf at a time!

unique choice!

Summary: Determinant-Based Counting

- Spanning trees: det(Laplacian$_{-\text{row},\text{col}}$)

  HW: directed case

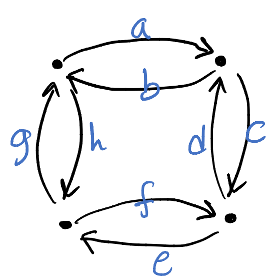
- Planar bipartite perfect matchings:

  det (signed adj. matrix)

  HW: non-bipartite case

- Directed Eulerian tours:

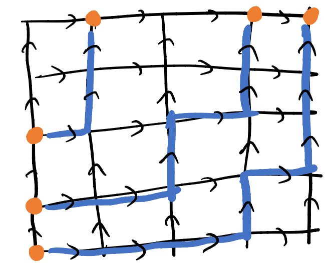
  Direct many-to-many correspondence to directed spanning trees.



[de Bruijn – van Aardenne-Ehrenfest
Smith – Tutte]

- acegh fdb
- abhfdceg
- ...

- Paths on DAGs



#(non-intersecting paths from left terminals to top terminals)

[Lindström – Gessel – Viennot]

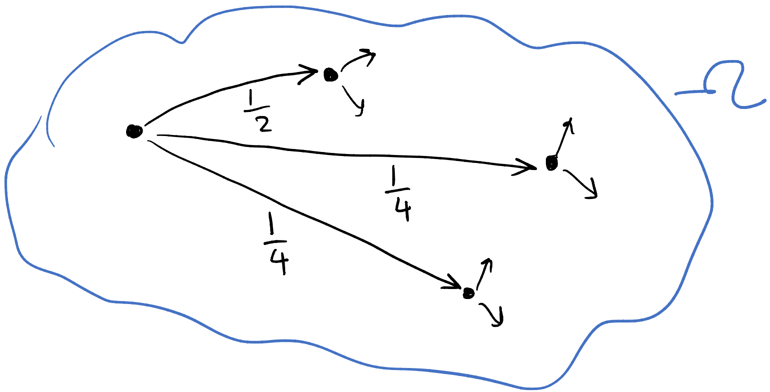
dynamic programing + determinants

- Determinantal Point Processes

  later in the course

- Holographic Reductions [Valiant]

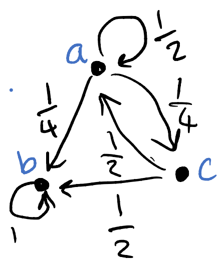
Intro to Markov Chains



Transition matrix $\underbrace{P \in \mathbb{R}_{\geq 0}^{\Omega \times \Omega}}_{\text{large \& implicit}}$

$\forall x: \sum_y P(x,y) = 1$ ← row-stochastic

Example.



$$
\begin{array}{c|ccc}
 & a & b & c \\
\hline
a & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
b & 0 & 1 & 0 \\
c & \frac{1}{2} & \frac{1}{2} & 0
\end{array}
$$

If we specify (random) $X_0 \in \Omega$, we get Markovian process: ← set of random variables

$$X_0 \xrightarrow{\text{transition via } P} X_1 \xrightarrow{\text{transition via } P} X_2 \to \cdots$$

Defining Characteristics:

- $\underbrace{\dots, X_{i-1}}, X_i, \underbrace{X_{i+1}, \dots}$ conditional on X_i, independent.
- X_{i+1} conditioned on X_i $\sim P$

Markov Chains $\Rightarrow$ approximate sampling

Fundamental Thm: Under mild conds:

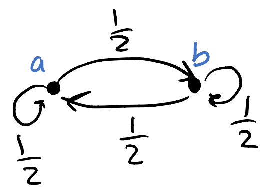
$$\text{dist}(X_t) \to \mu$$ ← stationary dist

Goals:
- Efficient transitions } easy
- Correct μ
- Fast convergence ← hard

Stationary Distribution

Suppose $X_0 \sim \nu$, then $X_1 \sim \nu P$
ν: row vector
P: matrix

Example.



If $X_0 = a$ then $\nu = [1 \quad 0]$

$$\nu P = [1 \quad 0] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\tfrac{1}{2} \quad \tfrac{1}{2}]$$

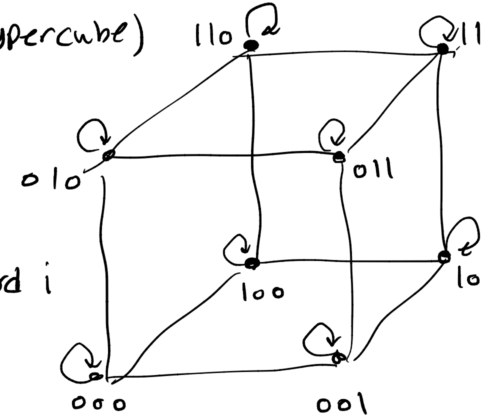
Stationary Dist: If $\mu P = \mu$ then μ is called stationary distribution.

Note: $\text{dist}(X_t) = \text{dist}(X_0) \cdot P^t$, so if it converges, it must converge to stationary!

Example. (Hypercube)

$\Omega = \{0,1\}^n$

- Pick random $i \in \{1, \ldots, n\}$
- Replace coord i by $Ber(\frac{1}{2})$

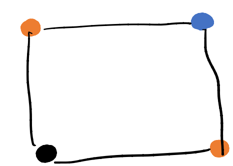


Stationary: uniform over Ω

Example. (Graph Coloring)

$\Omega = \{\text{valid colorings}\}$

- Pick random v unif.
- Recolor v unif from valid colors



stationary: uniform over Ω

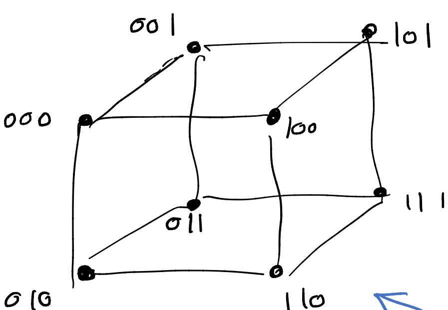
$$\mathbb{P}[X_1] = \frac{1}{n} \sum_{v} \sum_{X_0 \sim_v X_1} \frac{1}{\#\text{ available colors}} \cdot \frac{1}{|\Omega|} = \frac{1}{|\Omega|}$$

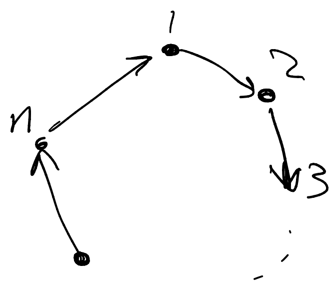
Fundamental Thm of Markov Chains

- Irreducibility:
  Possible to reach from every x to every y with >0 prob.

- Aperiodicity:
  For large enough t, possible to cycle $x \to x$ in exactly t steps. $\gcd(\text{lengths}) = 1$

$\Big\}$ ergodic





periodic

Thm: For an ergodic chain

$\exists$ unique stationary dist μ and for any starting X_0

$$\lim_{t\to\infty} d_{TV}(X_t, \mu) = 0$$

Remark: No assumption $\Rightarrow \exists$ stationary dist

Proof: P is a map from $\Delta(\Omega) \to \Delta(\Omega)$

$\Delta(\Omega) = \{\text{prob dists on } \Omega\}$.

$\Rightarrow$ must have ≥ 1 fixed points.

(e.g. by Brouwer)

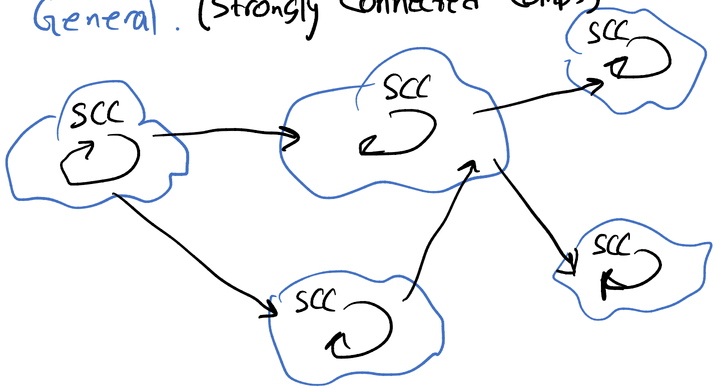
Remark: Irreducible $\Rightarrow$ Unique stationary

Proof: Apply Fundamental thm for $\frac{(I+P)}{2}$

Example. (Reducible)

Stationary: any distribution

Most General. (Strongly Connected Comps)



- For each terminal SCC, we get one stationary dist.
- Convex combinations of above = all stationary dists.

Proof of Fundamental Thm

Attempt #1:

Lemma: $d_{TV}(\nu P, \nu' P) \leq d_{TV}(\nu, \nu')$

If we could sneak 0.99 we'd be done.

- $\nu, \nu P, \nu P^2, \nu P^3, \dots$ ← Cauchy sequence

$$d_{TV}(\nu P^n, \nu P^m) \leq 0.99^n \cdot (1 + 0.99 + 0.99^2 + \dots) d_{TV}(\nu, \nu P)$$

So it converges.

- Stationary is unique:

$$d_{TV}(\mu P, \mu' P) \leq 0.99\, d_{TV}(\mu, \mu').$$

Bad Example.