Review

$\#P$: number of accept paths in nondet TM.

$\#P$-complete:

- counting version of all NP-complete problems we know via parsimonious reds.
- counting versions of some P probs:

  DNF, bipartite PM, stable matching

Approx Counting

FPTAS / FPRAS

$(1+\varepsilon)$-approx in $\mathrm{poly}(n, \frac{1}{\varepsilon})$

Approx Sampling

FPAUS

δ-d_{TV} approx $\mathrm{poly}(n, \lg \frac{1}{\delta})$

DNF Sampling/Counting:

- Rejection sampling
- Monte Carlo Estimation

Plan for Today

- Finish DNF Counting
- Prove:

$$\begin{tikzcd}
\text{Exact Counting} \arrow[r] \arrow[d] & \text{Approx Counting (FPRAS)} \arrow[d, leftrightarrow, "{[\text{Jerrum-Valiant-Vazirani}]}"] \arrow[dl, dashed, "\text{if FPTAS}"'] \\
\text{Exact Sampling} \arrow[r] & \text{Approx Sampling (FPAUS)}
\end{tikzcd}$$

- Exact counting via dets

if time

DNF Counting

$$\Phi = C_1 \vee C_2 \vee \cdots \vee C_m$$

$$A_i = \{ x \in \{0,1\}^n \mid x \text{ sats } C_i \}$$

Idea: – Estimate $\dfrac{|A_1 \cup \cdots \cup A_m|}{|A_1 \sqcup \cdots \sqcup A_m|}$

and multiply by $|A_1 \sqcup \cdots \sqcup A_m|$.

– We have an alg which accepts w. prob $p = |A_1 \cup \cdots \cup A_m| / |A_1 \sqcup \cdots \sqcup A_m|$.

– Run it t times and output

$$\frac{Ber(p) + Ber(p) + \cdots + Ber(p)}{t}$$

$X_i = Ber(p)$ $\quad X = \dfrac{X_1 + \cdots + X_t}{t}$.

$\mathbb{E}[X] = p$ $\quad Var(X) = \dfrac{t \cdot p(1-p)}{t^2} \leq \dfrac{p}{t}$.

Chebyshev's Inequality:

$$\mathbb{P}[X \notin [p - \varepsilon p, p + \varepsilon p]] < \frac{p/t}{(\varepsilon p)^2} = \frac{1}{t \cdot p \cdot \varepsilon^2}$$

Enough to let $t > 3/p\varepsilon^2$ to guarantee success w. prob $\geq 2/3$.

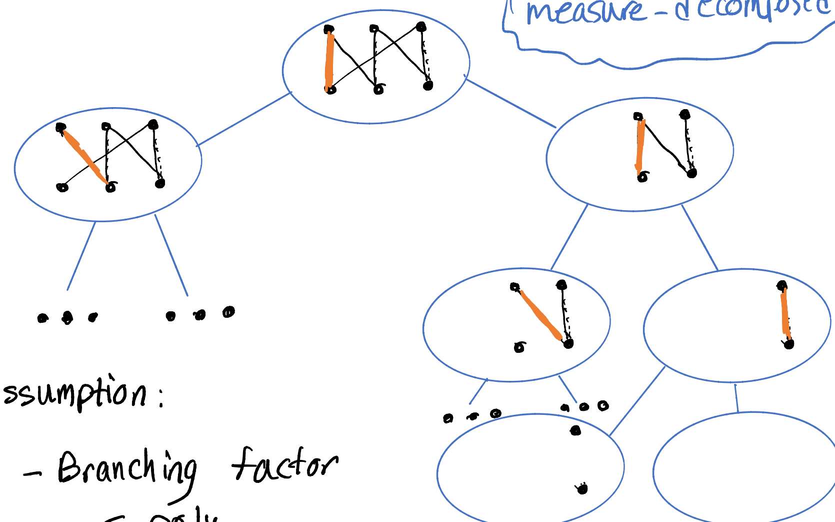
Take away: multiplicatively

To estimate p from $Ber(p)$ samples $\simeq O\left(\dfrac{1}{p\,\varepsilon^2}\right)$ samples enough.

Self-Reducible Problems

Informal Def: Solutions y to instance x can be partitioned into subsets in correspondence with instances x' of the same problem.

More advanced: measure-decomposed



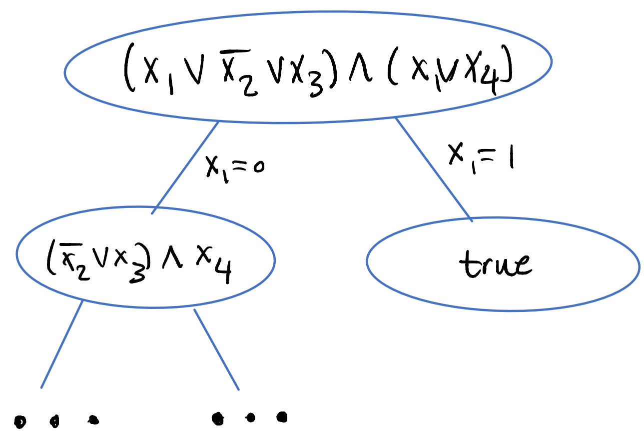
Assumption:

- Branching factor $\leq$ poly
- Depth $\leq$ poly

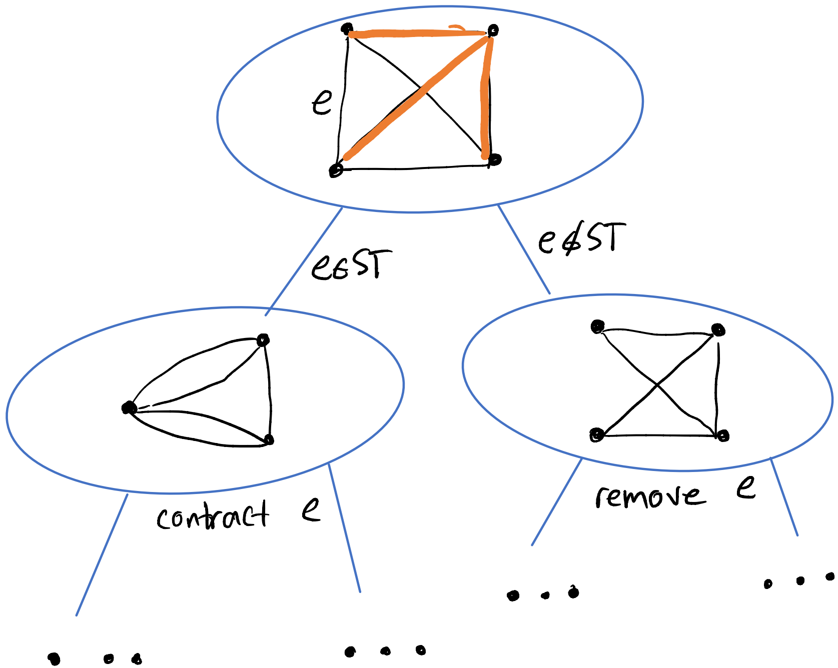
- Formal Requirements:

  * x's produced efficiently
  * one-to-one correspondence efficiently computable
  * There is a size $f(x)$ s.t. $f(x') < f(x)$ and $f(x) = \text{poly}(|x|)$
  * Base cases need no partitioning.

Example. (SAT)



Example. (Spanning Trees)

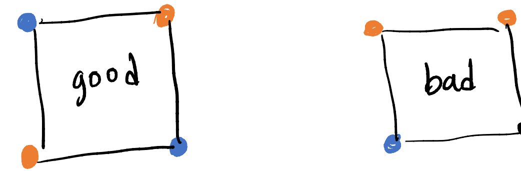


Non-Example. (Graph Coloring)

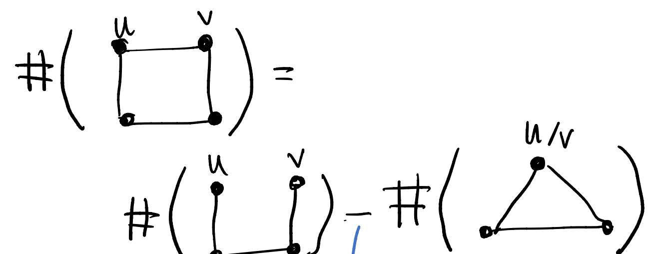
Graph $G=(V,E)$ and $q>0$:

$$\Omega = [q]^V$$

$$\mu(\omega) = 1[\text{no adjacent verts have same color}]$$

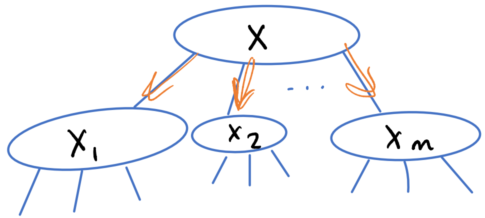


Note that



but this is NOT self-reducibility.

Exact Counting $\Longrightarrow$ Exact Sampling



Alg: - $X \leftarrow$ root

- Compute $\mu(x_i) = \sum_{y \in \text{part } i} \mu(y)$
- Choose i w.p. $\propto \mu(x_i)$
- $X \leftarrow X_i$ and repeat until base case.
- Move base-case sol. through one-one correspondences

$$\Pr[\text{leaf}] = \frac{\cancel{\mu(X_1)}}{\mu(X)} \cdot \frac{\cancel{\mu(X_{1,2})}}{\cancel{\mu(X_1)}} \cdot \,\cdots\, \cdot \frac{\mu(\text{leaf})}{\cancel{\mu(\text{parent})}}$$

Approx Det. Counting $\Longrightarrow$ Exact Sampling

What happens if we use $\tilde{\mu}(x_i)$, an approximation to $\mu(x_i)$?

$$\Pr_{\text{ALG}}[\text{leaf}] \overset{(1+\varepsilon)^{\text{depth}}}{\simeq} \frac{\mu(\text{leaf})}{\mu(X)}$$

$\leftarrow$ multiplicative error

If we let $\varepsilon \simeq \frac{1}{\text{depth}}$, then

$$\nu = \Pr_{\text{ALG}}[\text{leaf}] \overset{e}{\simeq} \frac{\mu(\text{leaf})}{\mu(X)}.$$

Idea: Use rejection sampling!

- We can compute $\Pr[\text{leaf}]$.
- Accept w.p. $\dfrac{\mu(\text{leaf}) / \tilde{\mu}(x)}{e^3 \times \Pr_{\text{ALG}}[\text{leaf}]}$.
- Repeat if rejected.

$$\Pr[\text{accept}] = \Omega(1).$$

Approx Rand. Counting $\Rightarrow$ Approx Sampling

Now there is chance of error in approx counting, but ...

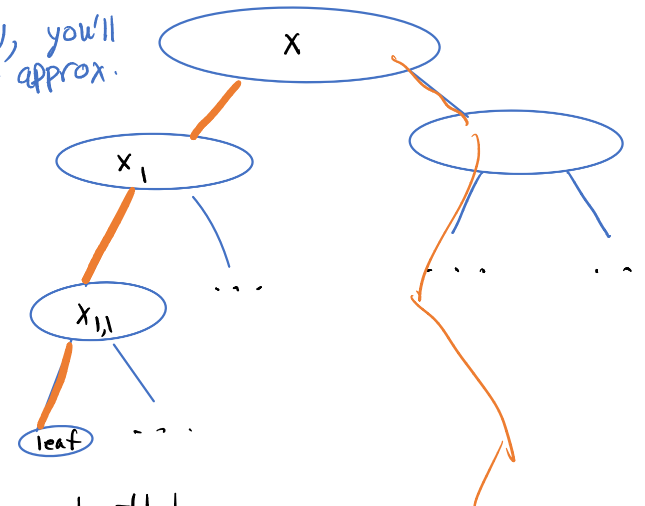
- Cut rejection sampling after $O(\lg \frac{1}{\delta})$ iterations. $\Rightarrow$ $\frac{\delta}{2}$ chance of not finishing.
- Total # approximate counts we need $\text{poly}(n) \cdot \lg \frac{1}{\delta}$.
- Make sure each succeeds w.p. $1 - \frac{1}{\text{poly}(n) \cdot \lg \frac{1}{\delta}} \cdot \frac{\delta}{2}$.

overhead $= \lg(\cdots) \leq \text{poly}(\lg n, \lg \frac{1}{\delta})$

overall runtime : $\text{poly}(n, \lg \frac{1}{\delta})$

Exact Sampling $\Longrightarrow$ Approx Counting

In HW, you'll show approx.



Attempt #1:

Pick root $\rightsquigarrow$ leaf path.

Estimate $\frac{\mu(x_1)}{\mu(x)}, \frac{\mu(x_{1,1})}{\mu(x_1)}, \ldots$

Multiply estimates to get

$\simeq \frac{\mu(\text{leaf})}{\mu(x)}$ ← computable

We need $1+\frac{\varepsilon}{2\cdot \text{depth}}$ approx in each step w. prob $\geq 1-\frac{1}{3\cdot \text{depth}}$.

$\Rightarrow$ multiplicative error $\leq \left(1+\frac{\varepsilon}{2\text{depth}}\right)^{\text{depth}} \leq 1+\varepsilon$

$\Rightarrow$ success prob $\geq 1-\frac{\text{depth}}{3\text{depth}} \geq \frac{2}{3}$

How do we estimate $\frac{\mu(X_1)}{\mu(X)}$?

Monte Carlo Estimation:

Take sample for X and see if it corresponds to X_1? $\Pr[\text{accept}] = \frac{\mu(X_1)}{\mu(X)}$.

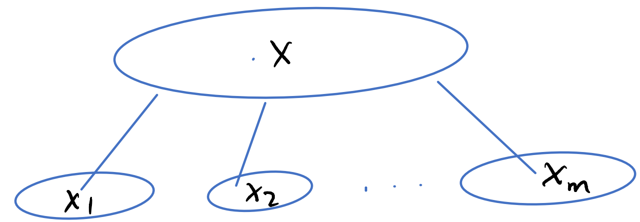
How many samples?

$$\frac{1}{\left(\frac{\varepsilon}{2\text{depth}}\right)^2 \cdot \frac{\mu(X)}{\mu(X_1)}} \cdot \lg(3\cdot \text{depth})$$

$\rightarrow$ potentially small

Attempt #2:

While $\frac{\mu(X_1)}{\mu(X)}$ could be small, for some i, $\frac{\mu(X_i)}{\mu(X)} \geq \frac{1}{m}$.



- Take a sample and see which X_i has it.
- Branch into that X_i and repeat until root $\rightsquigarrow$ leaf completed.

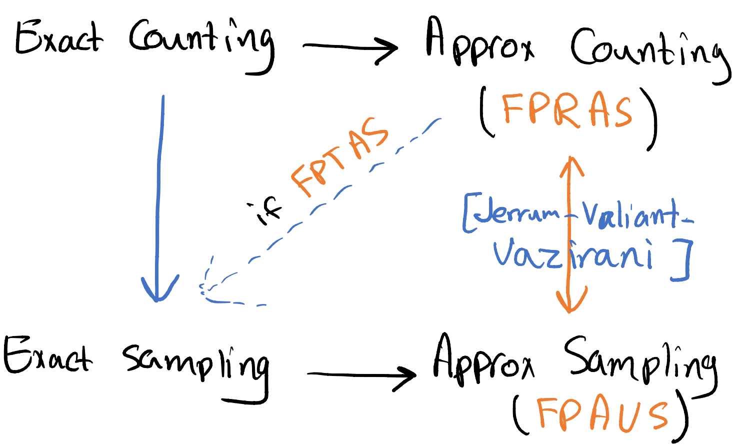
$$\mathbb{P}\left[\frac{\mu(X_i)}{\mu(X)} < \frac{\text{poly}(\varepsilon)}{\text{poly}(n)}\right] < \frac{\text{poly}(\varepsilon)}{\text{poly}(n)}$$

$$\Downarrow$$

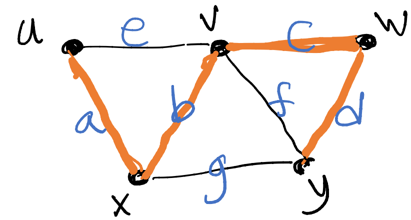
We can estimate w.h.p. using $\text{poly}(n, \frac{1}{\varepsilon})$ samples

For "self-reducible" problems

Counting $\equiv$ Sampling



Spanning Trees [Matrix-Tree Thm of Kirchhoff]



Vertex-Edge Adj Matrix:

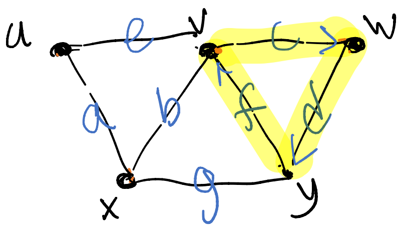
$$
\begin{array}{c|ccccccc}
 & a & b & c & d & e & f & g \\
\hline
u & +1 & 0 & 0 & 0 & +1 & 0 & 0 \\
v & 0 & -1 & +1 & 0 & -1 & -1 & 0 \\
w & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\
x & -1 & +1 & 0 & 0 & 0 & 0 & -1 \\
y & 0 & 0 & 0 & -1 & 0 & +1 & +1
\end{array}
$$

$n \times m$ matrix.

sum of all rows $= 0 \implies$ rank $\leq n-1$

rank $\leq n-1 \implies$ $n \times n$ submatrices have 0 det

What about $(n-1)\times(n-1)$ submatrices?



Vertex-Edge Adj Matrix:

$$\begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline u & +1 & 0 & 0 & 0 & +1 & 0 & 0 \\ v & 0 & -1 & +1 & 0 & -1 & -1 & 0 \\ w & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ x & -1 & +1 & 0 & 0 & 0 & 0 & -1 \\ y & 0 & 0 & 0 & -1 & 0 & +1 & +1 \end{array}$$

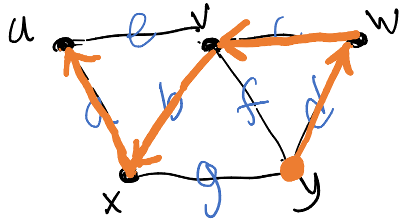
① If edges (columns) form cycle

$\implies$ det $= 0$

② Else they must form spanning tree

$\implies$ det $\in \{\pm 1\}$

[sketch on next slide]



$$\begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline u & +1 & 0 & 0 & 0 & +1 & 0 & 0 \\ v & 0 & -1 & +1 & 0 & -1 & -1 & 0 \\ w & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ x & -1 & +1 & 0 & 0 & 0 & 0 & -1 \\ y & 0 & 0 & 0 & -1 & 0 & +1 & +1 \end{array}$$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline u & +1 & 0 & 0 & 0 \\ v & 0 & -1 & +1 & 0 \\ w & 0 & 0 & -1 & +1 \\ x & -1 & +1 & 0 & 0 \end{array}$$

$+$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline u & +1 & 0 & 0 & 0 \\ v & 0 & -1 & +1 & 0 \\ w & 0 & 0 & -1 & +1 \\ x & 0 & +1 & 0 & 0 \end{array}$$

$+$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline u & +1 & 0 & 0 & 0 \\ v & 0 & 0 & +1 & 0 \\ w & 0 & 0 & -1 & +1 \\ x & 0 & +1 & 0 & 0 \end{array}$$

$+$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline u & +1 & 0 & 0 & 0 \\ v & 0 & 0 & +1 & 0 \\ w & 0 & 0 & 0 & +1 \\ x & 0 & +1 & 0 & 0 \end{array}$$

$$\det = \pm 1$$

How to sum?

Cauchy-Binet:

$$\det\left(\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} B \end{bmatrix}\right) = \sum_S \det\left(A_{cols=S}\, B_{rows=S}\right)$$

Let $A = B^T =$ adj with one row removed!

$\det(AA^T) = \#$ spanning trees