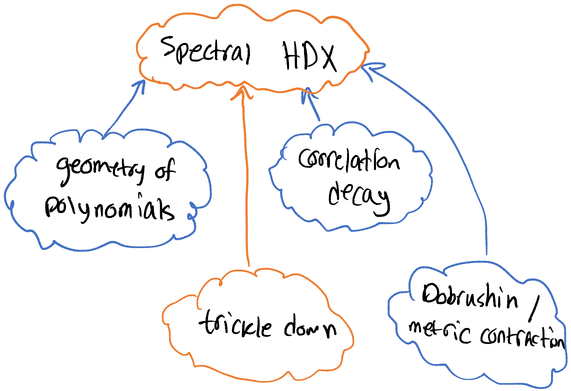
Trickle Down & Matroids



- We showed spanning trees are good HDX $\Rightarrow$ DU walk mixes in $\tilde{O}(|\text{verts}|)$ time.

- How about k-forests?
  forests with k edges

even poly-time sampling was open till 2019!

- We will show k-forests are as good of an HDX as spanning trees ($k = |\text{verts}| - 1$)

Remark: For k-forests, we no longer have g_μ is half-plane stable!

Matroids

$([n], \mathcal{I} \subseteq 2^{[n]})$

independent sets

- Axiom 1: $I \in \mathcal{I}$, $J \subseteq I \Rightarrow J \in \mathcal{I}$
- Axiom 2: $I, J \in \mathcal{I}$, $|J| > |I| \Rightarrow$

  $\exists e \in J - I$ s.t. $I \cup \{e\} \in \mathcal{I}$

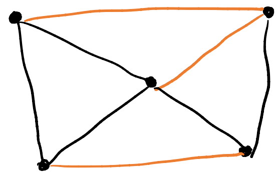
Note: Maximal sets have same size k.
We call them bases.

called rank

Example (Graphic)

$[n]$ = edges

$\mathcal{I}$ = acyclic subsets



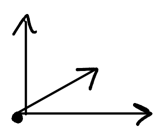
Axiom 1 ✓

Axiom 2: $|J| > |I| \Rightarrow$ some edge of J must stick out of a CC of I.

Example (Linear)

$[n]$ = vectors

$\mathcal{I}$ = lin. ind. subsets



$v_1, \ldots, v_n \in \mathbb{F}^d$

Axiom 1 ✓

Axiom 2: $|J| > |I| \Rightarrow$ some vector in J must stick out of span I.

Lem: If we truncate matroid to sets of size $\leq t$ for some t, we still have matroid :)

Corollary: $\exists$ matroid whose bases are k-forests of a graph.

Thm: If μ is uniform over bases of a matroid $\Rightarrow g_\mu$ 1-lg-concave

very good HDX

[A-Lin-OveisGharan-Vinzant]

[Brändén-Huh]

Corollary: $D_{KL}(\nu \| \mu) \geq k \cdot D_{KL}(\nu D_{k\to 1} \| \mu D_{k\to 1})$

Corollary: DU walk mixes in $\tilde{O}(k)$ time.

Strategy:

μ_T = dist of $S-T$ if $S\sim\mu$ cond $\supseteq T$

- Prove for $\underline{k=2}$ $\quad \underline{|T|}=k-2$
- Use trickle-down for general k

Trickle-Down (Informal): If $\mu_1, \dots, \mu_n$ (links) have χ^2-contraction $\Longrightarrow$ μ has χ^2-contraction.

$D_{\cdot\to 1}$ operator

Remark: If μ is (unif. over bases of) a matroid, so are its links μ_T.

Proof: Trivial to check axioms. $\square$

Trickle Down [Oppenheim]

Let μ be dist on $\binom{[n]}{k}$ and $\mu^1, \dots, \mu^n$ be $S\sim\mu$ cond on $1\in S, \dots, n\in S$ respectively.

Let $p = \mu D_{k\to 1}$. Then

$$\mu = \underbrace{p_1\mu^1 + \dots + p_n\mu^n}_{\text{decomposition of measure}}$$

Let us relate $\operatorname{cov}(\mu) = \mathbb{E}_{S\sim\mu}[1_S 1_S^T] - \mathbb{E}[1_S]\mathbb{E}[1_S]^T$ to $\operatorname{cov}(\mu_1), \dots, \operatorname{cov}(\mu_n)$.

We have

$$\mathbb{E}_\mu[1_S] = p_1\mathbb{E}_{\mu^1}[1_S] + \dots + p_n\mathbb{E}_{\mu^n}[1_S]$$
$$\mathbb{E}_\mu[1_S 1_S^T] = p_1\mathbb{E}_{\mu^1}[1_S 1_S^T] + \dots + p_n\mathbb{E}_{\mu^n}[1_S 1_S^T]$$

$$\mathrm{cov}(\mu) - \mathbb{E}_{i\sim p}[\mathrm{cov}(\mu^i)] =$$

$$\sum_i p_i \mathbb{E}_{\mu^i}[1_s]\mathbb{E}_{\mu^i}[1_s]^T - \mathbb{E}_{\mu}[1_s]\mathbb{E}_{\mu}[1_s]^T$$

Claim: This is $\frac{1}{k^2}\mathrm{cov}(\mu)\,\mathrm{diag}(p)^{-1}\mathrm{cov}(\mu)$!

Proof: Let $M = \mathbb{E}_{\mu}[1_s 1_s^T]$. Then
symmetric.

$$\mathbb{E}_{\mu^i}[1_s] = \frac{M_i}{k p_i}.$$

colum/row i

So we have

$$\sum_i p_i \mathbb{E}_{\mu^i}[1_s]\mathbb{E}_{\mu^i}[1_s]^T = \sum p_i \frac{M_i}{kp_i}\cdot\frac{M_i^T}{kp_i} =$$

$\frac{1}{k^2} M\,\mathrm{diag}(p)^{-1} M$. We now compute

$$\mathrm{cov}(\mu) = M - k^2 pp^T \implies$$

$$\frac{1}{k^2}\mathrm{cov}(\mu)\,\mathrm{diag}(p)^{-1}\mathrm{cov}(\mu) =$$

$$\frac{1}{k^2} M\,\mathrm{diag}(p)^{-1} M^T - \underbrace{p\,p^T\mathrm{diag}(p)^{-1} M}_{k^2 pp^T}$$

$$- \underbrace{M\,\mathrm{diag}(p)^{-1}}_{k^2 p} pp^T + k^2 p \underbrace{p^T\,\mathrm{diag}(p)^{-1} p}_{=1} p^T$$

$$= \frac{1}{k^2} M\,\mathrm{diag}(p)^{-1} M^T - \underbrace{k^2 pp^T}_{= \mathbb{E}_{\mu}[1_s]\mathbb{E}_{\mu}[1_s]^T}$$

$\square$

So we have the recurrence

$$\mathrm{cov}(\mu) = \mathbb{E}_{i\sim p}[\mathrm{cov}(\mu^i)] + \frac{\mathrm{cov}(\mu)\,\mathrm{diag}(p)^{-1}\mathrm{cov}(\mu)}{k^2}$$

Now suppose each μ^i is a good HDX:

$$\mathrm{cov}(\mu^i) \preceq C\,\mathrm{diag}\left(\mathbb{E}_{S\sim\mu^i}[1_{S-i}]\right)$$

row i = col i = 0

Important: we can drop i

Then we get

$$\mathrm{cov}(\mu) \preceq C\cdot \mathbb{E}_{i\sim p}\left[\mathrm{diag}\left(\mathbb{E}_{\mu^i}[1_{S-i}]\right)\right] + \frac{\mathrm{cov}(\mu)\,\mathrm{diag}(p)^{-1}\,\mathrm{cov}(\mu)}{k^2}$$

$$\mathbb{E}_{i\sim p}\left[\mathrm{diag}\left(\mathbb{E}_{\mu^i}[1_S]\right)\right] = k\,\mathrm{diag}(p)$$

$$\Rightarrow \mathbb{E}_{i\sim p}\left[\mathrm{diag}\left(\mathbb{E}_{\mu^i}[1_{S-i}]\right)\right] = (k-1)\,\mathrm{diag}(p)$$

So if we call

$$X = \frac{1}{k}\,\mathrm{diag}(p)^{-\frac{1}{2}}\,\mathrm{cov}(\mu)\,\mathrm{diag}(p)^{-\frac{1}{2}}$$

Then

$$X \preceq \frac{k-1}{k}\,C\cdot I + \frac{X^2}{k}$$

Note that eigenvalues of X satisfy the same inequality.

For $C=1$, this means $\lambda \leq \frac{k-1}{k} + \frac{\lambda^2}{k}$

$\Rightarrow \lambda$ is either ≤ 1 or $\geq k-1$.

this means disconnect

cannot happen for matroids $\longrightarrow$ $\mu = \mu' + \mu''$

different ground sets

$$\frac{1}{2}\mu' + \frac{1}{2}\mu''$$

Conclusion: If $C=1$ for links & "no disconnect" then $C\le 1$ for μ.

Remark: For larger C, the bound we get for μ is worse than the bound for links.

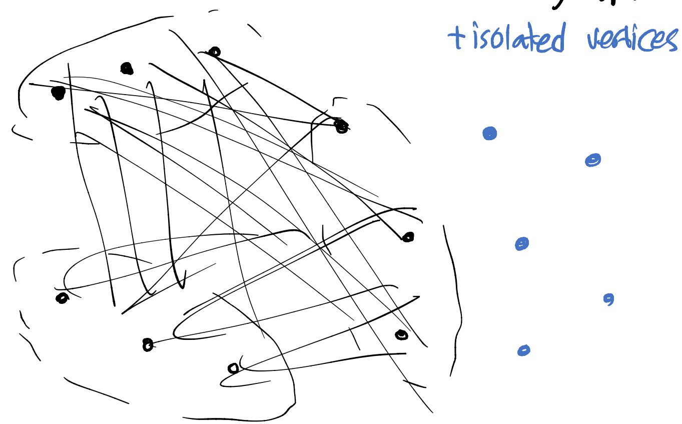
Open: Can we make it lossless in certain settings beyond matroids?

Since we can go from links to μ

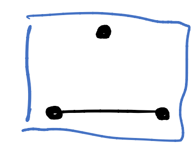
$\Rightarrow$ enough to show top links μ_T for $|T| = k-2$ are good HDX.

Question: Why are matroids of rank 2 good HDX?

Answer: They are complete multipartite graphs.



Proof: Ignore isolated verts. Forbidden config:



By axiom 2.

This means "not having edge" is an equivalence relationship.

Equivalence classes: parts of partite graph.

So $g_\mu(z) \propto z^T A z$ where $A = \text{adj}$.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & \boxed{0} & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Claim: $\lambda_2(A) \le 0$.

Proof:

$$A = \underbrace{\begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}}_{\text{rank } 1} - \underbrace{\begin{bmatrix} 1 & & 0 \\ & \boxed{1} & \\ 0 & & 1 \end{bmatrix}}_{\preceq 0}$$

$\square$

Claim: $g_\mu(z)$ is half-plane-stable & thus lg-concave.

Proof: Suppose $(u+iv)^T A (u+iv) = 0$ with $u \in \mathbb{R}^n_{>0}$. Then

$$u^T A u = v^T A v \quad \text{and} \quad u^T A v = 0$$

Consider the 2×2 matrix

$$B = \begin{bmatrix} u^T A u & u^T A v \\ v^T A u & v^T A v \end{bmatrix} = \begin{bmatrix} u^T \\ v^T \end{bmatrix} A \begin{bmatrix} u & v \end{bmatrix}$$

- B has at most ≤ 1 positive eig.
- B has at least ≥ 1 positive eig.

$\Rightarrow \det(B) \le 0 \Rightarrow (u^T A u)(v^T A v) \le (u^T A v)^2$. ※

$\square$

So we proved that if μ is a matroid μ_T for "top links T" are half-plane-stable & thus 1-lg-concave.

$\Rightarrow$ By trickle down

$$\mathrm{cov}(\mu) \preceq \mathrm{diag}(\mathrm{mean}(\mu))$$

How about full lg-concavity?

We need to show for $\lambda \in \mathbb{R}^n_{>0}$ if we look at $\lambda * \mu$:

$$\lambda * \mu(S) \propto \left(\prod_{i \in S} \lambda_i\right) \mu(S)$$

It also has

$$\mathrm{cov}(\lambda * \mu) \preceq \mathrm{diag}(\mathrm{mean}(\lambda * \mu))$$

But note that $(\lambda * \mu)_T = \lambda' * \mu_T$.

Half-plane-stable $\Rightarrow$ half-plane-stable

So we can apply trickle down to $\lambda * \mu$.

So we just proved:

Thm: Matroids are 1-lg-concave.

Coupling from the Past

We saw techniques for det counting

$\Longrightarrow$ exact sampling!

Question: Can we use Markov chains to sample perfectly?

[Propp–Wilson]: Use coupling from the past.

Note that we cannot stop a chain at a deterministic time & hope we are fine.

Idea: What if we "pretend" chain has been running for a really long time & we just compute current state without simulating all history?

Def (Grand Coupling)

Suppose P is a Markov chain on Ω.

A distribution π on functions $f: \Omega \to \Omega$ is a grand coupling if $\forall x, y$

$$\mathbb{P}_{f \sim \pi}[f(x) = y] = P(x, y)$$

Note that f itself is deterministic once we sample it.

Example (Coloring):

We sample v, c and let f take configuration σ to σ' where

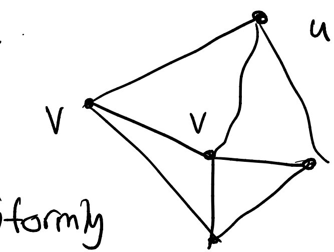
$$\sigma'(w) = \sigma(w) \quad \forall w \neq v$$
$$\sigma'(v) = \begin{cases} c & \text{if } c \text{ is valid} \\ \sigma(v) & \text{if not} \end{cases}$$

Once we sample v, c, f is fixed & deterministic.

Grand coupling for Metropolis

Example (ferro Ising)

Let μ on $\{\pm 1\}^n$ be $\propto \beta^{\sum_{u \sim v} x_u x_v}$, for some $\beta > 1$.



We sample v uniformly & $\tau \in [0,1]$ uniformly. Let X_{+1}, X_{-1} be configs where X_v is replaced with $+1, -1$ resp.

f maps X to either X_{+1} or X_{-1} based on

$$\tau \leq \frac{\mu(X_{+1})}{\mu(X_{+1}) + \mu(X_{-1})}$$

Grand coupling for Glauber.

Coupling from the past:

Let us sample i.i.d.

$f_{-1}, f_{-2}, f_{-3}, \ldots$ from

grand coupling and form

$g_T = f_{-1} \circ f_{-2} \circ \cdots \circ f_{-T}$. If

$g_T(\Omega)$ is a singleton we

we call this coalescence

output it.

Note: $g_T(\Omega) = \{x\} \to g_{T+1}(\Omega) = \{x\}$,

so the time T doesn't have to be the first.

Note: The last property is why we go to the past & not fwd.

Thm: Suppose coalescence happens w.p. 1.

Then the output follows stationary dist.

Proof: Note that $f_{-2} \circ f_{-3} \circ \cdots (\Omega)$ is identically distributed to the output of the alg.

(it's just shift by one).

So if X is this singleton we have

$f_{-1}(X)$ is identically distributed as X.

The dist of $X = \nu \Rightarrow \nu = \nu P$

$\square$

Although coupling from the past is very neat it is hard to check coalescence.

(Ω is exp. large)

But there are tricks:

Monotone coupling: Take grand coupling for Glauber dynamics on ferro Ising.

Exercise: Because of ferro all f are monotone: $x \geq y \Rightarrow f(x) \geq f(y)$.

So to check coalescence, we simply need to check

$$g_T(+1, \dots, +1) = g_T(-1, \dots, -1)$$

Thm: In $t_{mix} \cdot \lg n$ we coalesce with prob $\geq \frac{1}{2}$.

This was your HW.