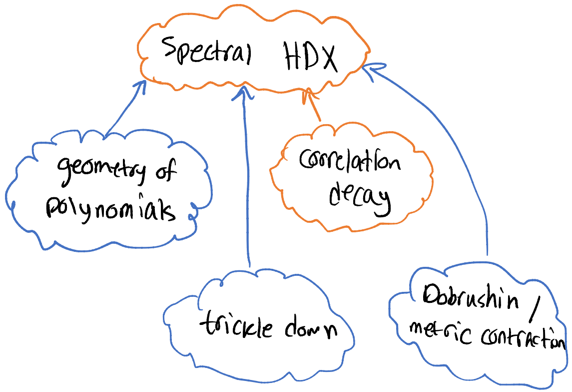
Methods for proving HDX



We now focus on correlation decay!

Main app: Show fast sampling of hardcore model for $\lambda < (1-\delta)\lambda_c(\Delta)$.

$\approx \frac{e}{\Delta}$

(Assume $\Delta = O(1)$)

Remark: We just need to show HDX.

Fast sampling follows because we have bounded deg spin system & lower bound on marginals:

$$\mathbb{P}_{X \sim \mu}[X(v)=0] \geq \frac{1}{1+\lambda} = \Omega(1)$$

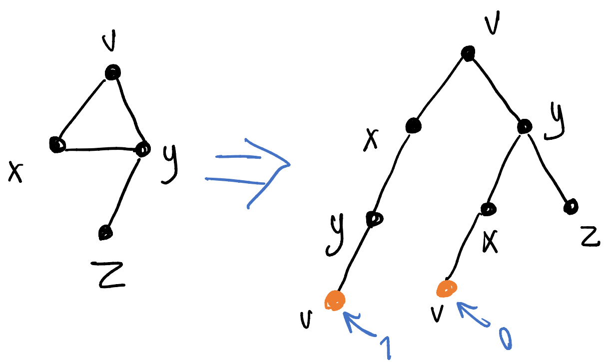
$$\mathbb{P}_{X \sim \mu}[X(v)=1] \geq \frac{\lambda}{1+\lambda}\left(\frac{1}{1+\lambda}\right)^{\Delta} = \Omega_{\Delta}(1).$$

Strategy [Chen-Liu-Vigoda]:

- Bound $\sum_j \left|\mathbb{P}[j \mid i] - \mathbb{P}[j \mid \bar{i}]\right|$ ← similar to root-freeness
- Relate above quantity on graphs to similar one on trees ← SAW tree

Reduction to Trees

We use the same strategy as in [Weitz] to show the following:



$$g_{graph} = \sum_{\text{ind } S} \prod_{i \in S} z_i \qquad g_{tree} = \sum_{\text{ind } S} \prod_{i \in S} z_{label(i)}$$

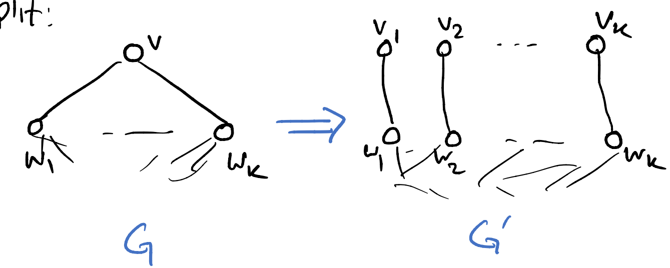
It generalizes to 2-spin systems.

Thm: $g_{tree} = g_{graph} \cdot h$ for some h. (has no z_{root})

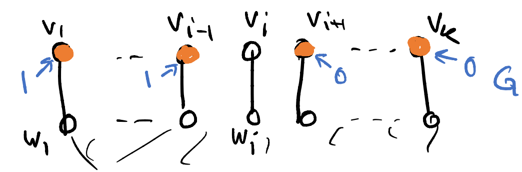
[Chen-Liu-Vigoda] [for matchings: Godsil]

Proof: We sketch it for hardcore but applies to all 2-spin systems.

Split:



Condition:



$$g_G = z_v \cdot g_G^{(1)} + g_G^{(0)}$$

$g_G^{(1)} \to g_{G'}(v_1, \dots, v_k \leftarrow 1)$

$g_G^{(0)} \to g_{G'}(v_1, \dots, v_k \leftarrow 0)$

$$g_{G_i} = z_v \cdot g_{G_i}^{(1)} + g_{G_i}^{(0)}$$

$g_{G_i}^{(1)} \to g_{G'}(v_1, \dots, v_k \leftarrow 11\dots10\dots0)$

$g_{G_i}^{(0)} \to g_{G'}(v_1, \dots, v_k \leftarrow 11\dots00\dots0)$

Note that $g^{(1)}_{G_i} = g^{(0)}_{G_{i+1}}$.

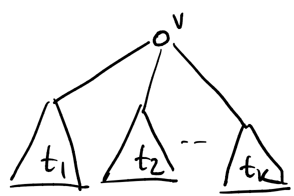
Let $q = g_{G'(v_1 \ldots v_k \leftarrow 01\ldots1)} \cdot g_{G'(v_1\ldots v_k \leftarrow 001\ldots1)} \cdots g_{G'(v_1 \ldots v_k \leftarrow 0\ldots0)}$

$$= g^{(1)}_{G_1} \cdots g^{(1)}_{G_{k-1}} = g^{(0)}_{G_2} \cdots g^{(0)}_{G_k}.$$

$$- \; q \cdot g_G = z_v \, g^{(1)}_{G_1} \cdots g^{(1)}_{G_{k-1}} g^{(1)}_{G_k} + g^{(0)}_{G_1} g^{(0)}_{G_2} \cdots g^{(0)}_{G_k}$$

What happens for tree:

$$g_{tree} = z_v \cdot \prod_i g^{(0)}_{t_i} + \prod_i g_{t_i}$$



By induction $\exists h_i$ s.t.

$$g_{t_i} = h_i \cdot g^{(0)}_{G_i}$$

doesn't have z_{w_i}

This means

$$g^{(0)}_{t_i} = h_i \, g^{(0)}_{G_i(w_i \leftarrow 0)} = h_i g^{(1)}_{G_i}$$

same as setting $v_i \leftarrow 1$

So we conclude

$$\underbrace{\left(q \cdot \prod h_i\right)}_{\text{our } h} g_G = z_v \prod_i \left(h_i g^{(1)}_{G_i}\right) + \prod_i \left(h_i g^{(0)}_{G_i}\right)$$

$$= z_v \prod_i g^{(0)}_{t_i} + \prod_i g_{t_i} = g_{tree}$$

$\square$

So we proved that $g_{tree} = g_{graph} \cdot h$.

Note that the root v appears just once, so h does not contain z_v or else the z_v-degrees wouldn't match.

We can now prove powerful things:

Lem: $\mathbb{P}_{graph}[v \text{ occ.}] = \mathbb{P}_{tree}[v \text{ occ.}]$

Proof: In general we can compute marginals as logarithmic derivatives.

$$\lambda \partial_{z_v} \lg g_{graph} \Big|_{z=\lambda 1} = \frac{\sum_{S:\text{ind}, S\ni v} \lambda^{|S|}}{\sum_{S:\text{ind}} \lambda^{|S|}} = \mathbb{P}_{graph}[v \text{ occ}].$$

Similarly $\mathbb{P}_{tree}[v \text{ occ.}] = \lambda \partial_{z_v} \lg g_{tree} \big|_{z=\lambda 1}$.

Note that $\lg g_{tree} = \lg g_{graph} + \underbrace{\lg h}_{\text{doesn't depend on } z_v}$

So the lg-derivatives are equal. $\square$

Another important quantity is

$$I[v \to u] = \mathbb{P}[u | v] - \mathbb{P}[u | \bar{v}]$$

don't confuse with Dobrushin

Ultimately our goal is to show

$$\forall v: \quad \sum_{v} |I[v \to u]| = O(1)$$

Lem: We have

$$I_{graph}[v \to u] = \sum_{u_i:\ u\text{'s copies in tree}} I[v \to u_i]$$

Proof: Note that we have

$$g_{tree} = g_{tree}^{(0)} + z_v g_{tree}^{(1)}$$

$$g_{graph} = g_{graph}^{(0)} + z_v g_{graph}^{(1)}$$

$$\Downarrow$$

$$g_{tree}^{(i)} = g_{graph}^{(i)} \cdot h \leftarrow \text{doesn't have } z_v.$$

Now it's easy to see

$$\mathbb{P}_{graph}[u|v] = \lambda \partial_{z_u} \lg g^{(1)}_{graph}\Big|_{z=\lambda}$$

$$\sum_i \mathbb{P}_{tree}[u_i|v] = \lambda \partial_{z_u} \lg g^{(1)}_{tree}\Big|_{z=\lambda}$$

This time the above are not equal, but we have

$$\lambda \partial_{z_u} \lg g^{(1)}_{tree} = \lambda \partial_{z_u} \lg g^{(1)}_{graph} + \underbrace{\lambda \partial_{z_u} \lg h}_{\text{extra term}}$$

By a similar calculation we have

$$\mathbb{P}_{graph}[u|\bar{v}] = \lambda \partial_{z_u} \lg g^{(0)}_{graph}\Big|_{z=\lambda}$$

$$\sum_i \mathbb{P}_{tree}[u_i|\bar{v}] = \lambda \partial_{z_u} \lg g^{(0)}_{graph}\Big|_{z=\lambda}$$

$$\lambda \partial_{z_u} \lg g^{(0)}_{tree} = \lambda \partial_{z_u} \lg g^{(0)}_{graph} + \overbrace{\lambda \partial_{z_u} \lg h}^{\text{extra term}}$$

When we subtract the extra terms cancel each other and we get

$$\sum_{u_i} \left(\mathbb{P}[u_i|v] - \mathbb{P}[u_i|\bar{v}]\right) = \mathbb{P}[u|v] - \mathbb{P}[u|\bar{v}]$$

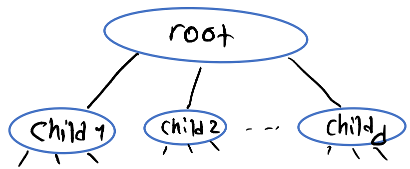
$\square$

So now to prove HDX we simply need to show $\sum_w |I_{tree}[v \to w]| = O(1)$

$$\sum_u |I_{graph}[v \to u]| = \sum_u \left|\sum_{u_i} I_{tree}[v \to u_i]\right| \leq \sum_w |I_{tree}[v \to w]|$$

Remember tree recursion:

p_v: $\mathbb{P}[x_v=0]$ in v's subtree



$$p_{root} = \frac{1}{1+\lambda\prod_i p_{child\ i}}$$

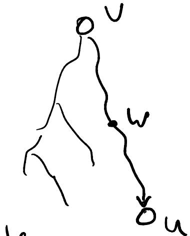
We reparameterize as $x_v = \lg \frac{1-p_v}{p_v}$

$$x_{root} = \lg\lambda - \sum_i \lg\left(1+e^{x_{child\ i}}\right)$$

$$= F\left(x_{child\ 1}, \dots, x_{child\ d}\right)$$

$$= F\left(F(\dots), F(\dots), \dots, F(\dots)\right)$$

Claim: $\frac{\partial x_v}{\partial x_u} = I[v \to u]$



Proof: First note that $I[v\to u]$ follows chain rule, the same way derivatives do.

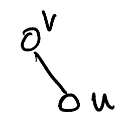
$$I[v\to u] = \mathbb{P}[u|v] - \mathbb{P}[u|\bar{v}] =$$

$$\mathbb{P}[w|v]\cdot\mathbb{P}[u|w] + \mathbb{P}[\bar{w}|v]\mathbb{P}[u|\bar{w}]$$
$$-\mathbb{P}[w|\bar{v}]\,\mathbb{P}[u|w] - \mathbb{P}[\bar{w}|\bar{v}]\mathbb{P}[u|\bar{w}]$$

$$= I[v\to w]\, I[w\to u]$$

So we just need to show it for one level:

$$I[v\to u] = \mathbb{P}[u|v] - \mathbb{P}[u|\bar{v}] = 0-(1-p_u)$$



$$= p_u - 1 = \frac{-e^{x_u}}{1+e^{x_u}} = \frac{\partial F(\dots, x_w, \dots)}{\partial x_u}$$

$\square$

This means

$$\sum_{u \text{ at level } i} \left| I[v \to u] \right| = \| \nabla F(\underbrace{F(-), \ldots, F(-)}_{i \text{ levels of nest}}) \|_1$$

If we had that $\|\nabla F\|_1 \leq 1-\delta$ then the above would be $\leq (1-\delta)^i$.

We could then sum over all levels:

$$\sum_u |I[v \to u]| \leq \sum_i (1-\delta)^i = O(1)$$

Unfortunately we don't have $\|\nabla F\| < 1-\delta$.

:(

We can use the same trick as before:

$$G := \psi \circ F(\psi^{-1}(x_1), \ldots, \psi^{-1}(x_d))$$

some change of variable

For some appropriate ψ, when $\lambda < (1-\varepsilon)\lambda_c(\Delta)$, we have $\|\nabla G\|_1 < 1-\delta$.

Moreover ψ is "nice": its derivatives have bounded aspect ratio.

$$\|\nabla F(F(--), \ldots, F(-))\|_1 \leq (1-\delta)^i \cdot \frac{\max \psi'}{\psi'(x_{root})}$$

$O(1)$

$\square$

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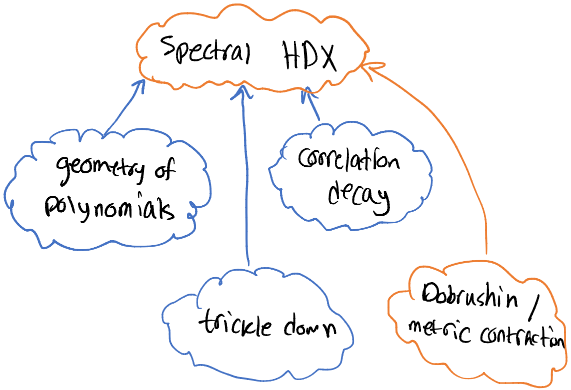

We now focus on Dobrushin.

This was proved by

[Liu], [Blanca-Caputo-Chen-Parisi-Štefankovič-Vigoda]

I will show a different argument

[A-Jain-Koehler-Pham-Vuong]

Suppose μ is a spin system on k verts.

$$\mu : \binom{\Omega_1 \sqcup \cdots \sqcup \Omega_k}{k} \to \mathbb{R}_{\geq 0}$$

From HW, we know that Dobrushin (or ++) implies $t_{rel} = O(k)$.

Thm: If $D_{k\to k-1}U_{k-1\to k}$ has $t_{rel} = O(k)$ then $D_{k\to 1}$ contracts χ^2 by $O(1/k)$.

Proof: We need to show

$$\mathrm{cov}(\mu) \preceq O(1)\cdot \mathrm{diag}(\mathrm{mean}(\mu))$$

which means for all $v \in \mathbb{R}^n$

$$v^T \mathrm{cov}(\mu) v \leq O(v^T \mathrm{diag}(\mathrm{mean}(\mu)) v)$$

Rewriting this:

$$v^T \mathrm{cov}(\mu) v = \mathbb{E}_{S\sim\mu}[(\sum_{i\in S} v_i)^2] - \mathbb{E}_{S\sim\mu}[\sum v_i]^2 \quad (1)$$

$$v^T \mathrm{diag}(\mathrm{mean}(\mu)) v = \mathbb{E}_{S\sim\mu}[\sum v_i^2] \quad (2)$$

So we need to show $(1) = O((2))$.

Note that $t_{rel} = O(k)$ implies $D_{k\to k-1}$ contracts χ^2 by $1 - \Omega(\frac{1}{k})$. We apply this to

$$\nu(S) = \mu(S)(\sum_{i\in S} v_i).$$

$$\chi^2(\nu \| \mu) = \underbrace{\mathbb{E}_{S\sim\mu}[(\sum_{i\in S} v_i)^2] - \mathbb{E}[\sum v_i]^2}_{(1)}$$

$$\chi^2(\nu \| \mu) - \chi^2(\nu D_{k\to k-1} \| \mu D_{k\to k-1}) =$$

$$\mathbb{E}_{T\sim\mu D_{k\to k-1}}\left[\mathbb{E}_{S\sim\mu|T}[(\sum_{i\in S} v_i)^2] - \mathbb{E}_{S\sim\mu|T}[\sum_{i\in S} v_i]^2\right]$$

$$= \mathbb{E}_{T\sim\mu D_{k\to k-1}}\left[\mathbb{E}_{i\sim\mu_T}[v_i^2] - \mathbb{E}_{i\sim\mu_T}[v_i]^2\right]$$

$$\leq \mathbb{E}_{T\sim\mu D_{k\to k-1}}\left[\mathbb{E}_{i\sim\mu_T}[v_i^2]\right] = \frac{1}{k}\underbrace{\mathbb{E}_{S\sim\mu}[\sum_{i\in S} v_i^2]}_{(2)}$$

So we conclude that

$$\frac{1}{k}(2) \geq \Omega\left(\frac{1}{k}\right)(1)$$

Which means $(1) \leq O((2))$ $\square$