Review

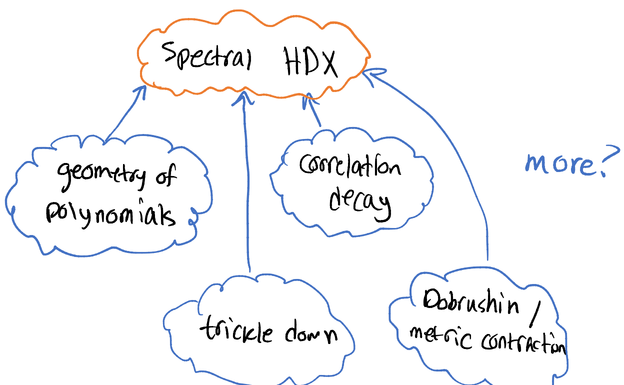
- Spectral HDX: $D_{k\to 1}$ contracts χ^2 by $\frac{C}{k}$

  * $\lambda_2\left[\,\mathbb{P}[j|i]\,\right]_{i,j} \leq C.$

  * $\lambda_{max}\left[\mathbb{P}[j|i]-\mathbb{P}[j]\right]_{i,j} \leq C.$

  * $\mathrm{cov}(\mu) \preceq C\cdot \mathrm{diag}(\mathrm{mean}(\mu)).$

  * $g_\mu(z_1^{\frac{1}{C}},\ldots,z_n^{\frac{1}{C}})$ lg-concave at $z=1$.



- Gårding: if $g_\mu \neq 0$ for $\mathrm{Re}(z_i)>0$

  $\Rightarrow g_\mu(z_1,\ldots,z_n)$ lg-concave. $\leftarrow C=1$

- Spanning trees & DPPs are HDX

$\Rightarrow$ down-up walk has relaxation time $O(k)$

Plan for Today:

- Monomer dist is HDX $\leftarrow C=2$
- Stability $\Rightarrow$ HDX
- From χ^2 to D_{KL}

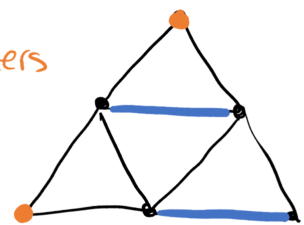
C-lg-concavity ← $C>1$

Example (Monomers)

Take a graph (possibly weighted) and the monomer-dimer system on it.

μ: dist of just monomers

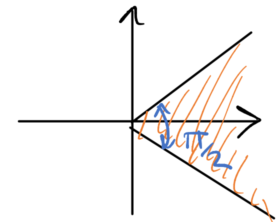
$\mu(S) \propto \#\text{PMs on } G-S$



Thm: [Heilmann-Lieb] The monomer dist has $g_\mu(z_1,\dots,z_n) = \mathbb{E}_{S\sim\mu}\left[\prod_{i\in S} z_i\right]$ stable in $\{z \mid \mathrm{Re}(z)>0\}$. not hom.

no longer equiv to all half planes

Corollary: Hom poly $z_1'\cdots z_n'\, g_\mu\left(\frac{z_1}{z_1'},\dots,\frac{z_n}{z_n'}\right)$ is stable in sector of aperture $\frac{\pi}{2}$:



$z^S \to z^S z'^{[n]-S}$

This is the generating poly of equiv dist on $\binom{V\times\{0,1\}}{|V|}$.

Corollary: Two-block dynamics on μ has spectral gap $\geq 1/|V|^2$. ← [A-Ivkov]

I will show 4-lg-concave instead of 2.

Application: Sampling/counting monomer-dimer on planar graphs. ← have oracle for μ.)

Proof of Heilmann-Lieb:

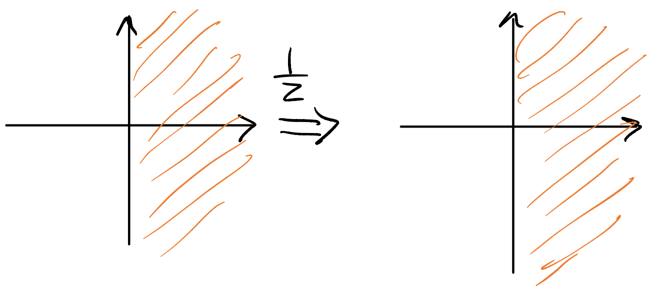
$$g = \mathbb{E}_{\text{matching } M}[z^{\text{monomers}(M)}]$$

only odd/even degree monomials

- Equivalently:

$$h = \mathbb{E}_{\text{matching } M}[z^{\text{matched}(M)}]$$

$$h = z_1 \cdots z_n \, g\left(\tfrac{1}{z_1}, \ldots, \tfrac{1}{z_n}\right)$$



- h has only even degree terms
- Suppose $h(a+ib)=0$ $a \in \mathbb{R}^n_{>0}$, $b \in \mathbb{R}^n$.

- This means $h(ta+ib)$ has root $t=1$.
- Claim: Roots of $h(ta+ib)$ are on imaginary axis.

$$q_G(t) := h_G(ta+ib)$$

Note: q takes real values on Im axis, because of even degree.

- Proof similar to real-rootedness of univariate matching polynomial:

$$q_G = \alpha q_{G-u} + \sum_{v \sim u} \beta_{uv} (ta_u + ib_u)(ta_v + ib_v) q_{G-u-v}$$

$(\alpha, \beta_{uv} > 0)$

Stronger Claim: Roots of q_G and $(ta_u + ib_u) q_{G-u}$ are imaginary and interlace.

$t = \frac{-b_u \cdot i}{a_u}$

q_{G-u}	$q_{G-u-v}(ta_u+ib_u)(ta_v+ib_v)$
$\vdots$	$\vdots$
0	$-/+$
0	$+/-$
0	$-/+$
$+/-$	0
0	$-/+$
0	$+/-$

$\exists$ root of q_G in between every two points

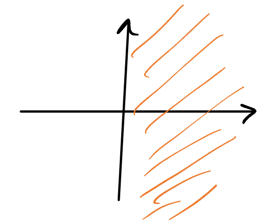
Exercise: top/bottom also have roots if $\deg(q_G) > \deg(q_{G-u})$.

blue dot : roots of q_{G-u}

orange dot : root of $(ta_u + ib_u)$, i.e., $\frac{-ib_u}{a_u}$

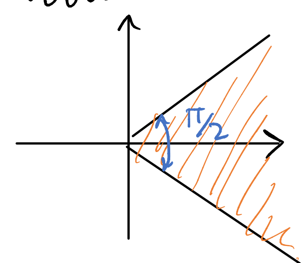
$\square$

This shows that g has no roots in



which means the generating poly of μ viewed on $\binom{V\times\{0,1\}}{|V|}$

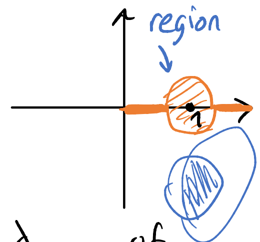
has no roots in



Remark: Alternatively we can look at monomer dist cond. on #monomers $= k$.

That dist is also $\frac{\pi}{2}$-sector stable & hence 2-lg-concave.

Thm: (Stability $\Rightarrow$ HDX)



We want to bound λ_{max} of

$$\text{Correlation matrix} \rightarrow \begin{bmatrix} \mathbb{P}[1|1]-\mathbb{P}[1] & \cdots & \mathbb{P}[n|1]-\mathbb{P}[n] \\ \vdots & \mathbb{P}[j|i]-\mathbb{P}[i] & \vdots \\ \mathbb{P}[1|n]-\mathbb{P}[1] & \cdots & \mathbb{P}[n|n]-\mathbb{P}[n] \end{bmatrix}$$

- We will show $\underbrace{\ell_1(\text{each row}) \leq O(1)}$.

this is the most common technique but often not tight

- Fact: $\lambda_{max} \leq \max\{\ell_1(\text{row}) \mid \text{row}\}$.

because it is $\infty \to \infty$ norm of the matrix

- We will show stronger:

$$\sum_j \left|\mathbb{P}[j|i] - \mathbb{P}[j|\bar{i}]\right| = O(1)$$

$\mathbb{P}[\cdot|\bar{i}] = \mathbb{P}[\cdot \mid i \notin S]$
Sum

This is stronger because

$$\forall j: \quad \mathbb{P}[j] = p\,\mathbb{P}[j|i] + (1-p)\,\mathbb{P}[j|\bar{i}]$$

$\mathbb{P}[i]$

- Let $g_\mu = z_i g_1 + g_0$ $\quad$ g_1, g_0 don't depend on z_i.

- Let $s_j = \text{sign}(\mathbb{P}[j|i] - \mathbb{P}[j|\bar{i}])$

- Now define

$$P_0, P_1 = g_0, g_1(z^{s_1}, z^{s_2}, \dots, z^{s_n})$$

If $z \in$ region $\Rightarrow z^{s_j} \in$ region

w.l.o.g. region closed under $z \mapsto z^{-1}$

- This means $\dfrac{P_1(z)}{P_0(z)} \notin \mathbb{R}_{\leq 0}$ $\quad$ $P_0 + z_i P_1 \neq 0$

- Claim:

$$\frac{P_1'(1)}{P_1(1)} = \sum_{j \neq i} s_j \mathbb{P}[j|i]$$

$$\frac{P_0'(1)}{P_0(1)} = \sum_{j \neq i} s_j \mathbb{P}[j|\bar{i}]$$

- Enough to show

$$\left| \frac{P_1'}{P_1} - \frac{P_0'}{P_0} \right| = O(1)$$

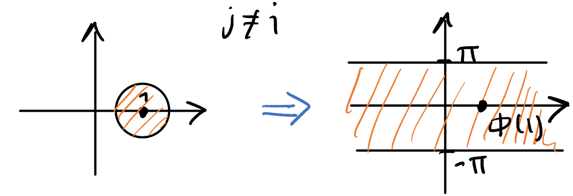
extra term for $j = i$ absorbed

- Since $\dfrac{P_1}{P_0} \notin \mathbb{R}_{\leq 0}$ in region, we can define a branch of lg:

$$\Phi(z) = \lg\left(\frac{P_1(z)}{P_0(z)}\right)$$

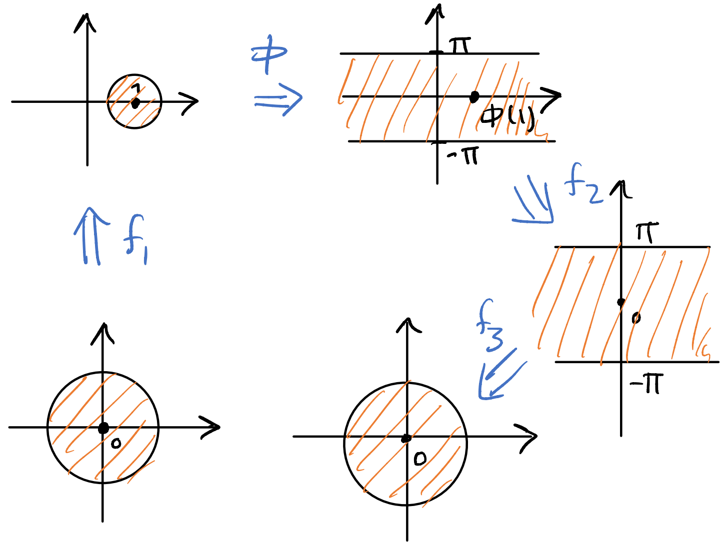
and $\Phi'(1) = \sum_{j \neq i} \left|\mathbb{P}[j|i] - \mathbb{P}[j|\bar{i}]\right|$

Φ :



|derivative at 1| < $O(1)$

for any such map (Schwartz Lemma)



$f_1 : z \longmapsto 1 + \alpha z$

$f_2 : z \longmapsto z - \Phi(1)$

$f_3 : z \longmapsto \dfrac{e^{z/2} - 1}{e^{z/2} + 1}$

$f_3 \circ f_2 \circ \Phi \circ f_1 : \text{Disk} \longmapsto \text{Disk}$

$\qquad\qquad\qquad\qquad 0 \longmapsto 0$

Schwartz Lemma: $\left| (f_3 \circ f_2 \circ \Phi \circ f_1)'(0) \right| \leq 1$

But this is

$$\underbrace{f_3'(0)}_{1/4} \cdot \underbrace{f_2'(\Phi(1))}_{1} \cdot \Phi'(1) \cdot \underbrace{f_1'(0)}_{\alpha}$$

$\Rightarrow \quad |\Phi'(1)| \leq O(1/\alpha) = O(1).$

Note: When domain of Φ is a sector we can find easier/tighter maps.

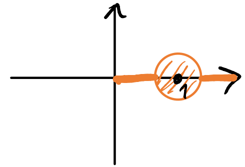
Proof of Schwartz: If f: Disk $\to$ Disk, $f(0) = 0$ ← holomorphic

then $|f'(0)| \leq 1$.

- Define $h(z) = f(z)/z$ for $z \neq 0$ and $h(0) = f'(0)$.
- h is holomorphic. (Taylor series shifted by 1)
- The max of $|h|$ obtained on boundary

$$|h(0)| \leq \max\{ |h(z)| \mid |z| = 1 \} \leq 1.$$

Summary: Stability in



$\Rightarrow$ HDX, i.e., contraction of χ^2 under $D_{k\to 1}$ for μ

Note: Also for links of μ because $\infty \in \overline{\text{region}}$ & links obtained by taking limits of $z_i \to \infty$.

Note: For sector of aperture $\frac{\pi}{c}$ the argument shows $O(c)$-lg-concavity.

$$g(z_1,\dots,z_n) \xmapsto{\forall \alpha \in \mathbb{R}^n_{>0}} g(\alpha_1 z_1,\dots,\alpha_n z_n)$$

$\downarrow$ stable in Disk $\cup\, \mathbb{R}_{>0}$

$\Downarrow$

$O(c)$-lg-concave at all z. $\Leftarrow$ $O(c)$-lg-concave at $z=1$

Note: Diff arg gives aperture π/c sector $\Rightarrow$ c-lg-concave [A-Ivkov].

Note: When region $\not\supseteq \mathbb{R}_{>0}$, [Chen-Liu-Vigoda] showed with extra assumptions on marginals of μ, argument still works.

$$\chi^2 \longrightarrow D_{KL}$$

So far we have studied spectral HDX. To get tight mixing one often needs entropic notions, like contraction of D_{KL}.

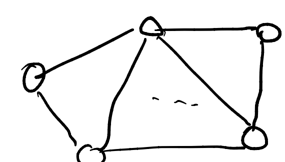
Q: Does χ^2 contraction imply D_{KL} contraction?

A: No without other assumptions.

Expander $\Rightarrow$ χ^2 contracts by constant factor

$\Rightarrow O(\lg n)$ mixing

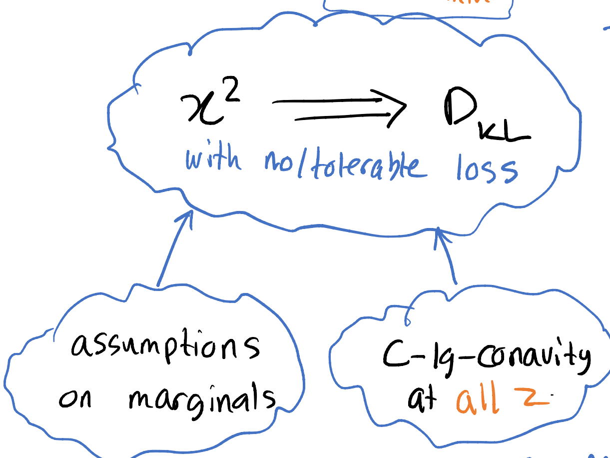
But $\lg n$ is tight!



Constant D_{KL} contraction would imply $O(\lg\lg n)$ mixing which is false. :(

Note: In general ρ-Poincare implies γ-LSI & thus γ-MLSI with

$\gamma \simeq \rho / \boxed{\lg\left(\frac{1}{\mu_{min}}\right)}$ ← this is useless for better t_{mix}



[Chen-Liu-Vigoda] based on above note

[Cryan-Guo-Mousa] (C=1)
[A-Jain-Koehler-Pham-Vuong] (general C)

Entropic Independence

What does D_{KL} contraction by $D_{k\to 1}$ mean?

- No longer a simple $n \times n$ matrix to analyze :(
- Fortunately still easy to describe via g_μ :)

Lem [A-Jain-Koehler-Pham-Vuong]

We have $D_{KL}(\nu D_{k\to 1} \| \mu D_{k\to 1}) \leq \frac{C}{k} D_{KL}(\nu \| \mu)$ if and only if $\lg g_\mu$ is upperbounded by its tangent at $z=1$:

$$\lg g_\mu(z^{\frac{1}{C}}) \leq \lg g_\mu(1) + \nabla \lg g_\mu(1) \cdot (z^{\frac{1}{C}} - 1)$$

$\forall z \in \mathbb{R}^n_{>0}$