Review

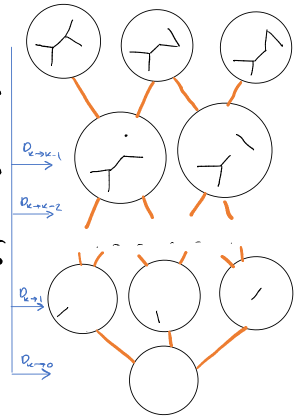
- Dist μ on $\binom{[n]}{k}$

- $D_{k\to\ell}(S,T) = \begin{cases} 0 & \text{if } T \not\subseteq S \\ \frac{1}{\binom{k}{\ell}} & \text{if } T \subseteq S \end{cases}$

  $\uparrow$ down operator

- $U_{\ell\to k}(T,S) = \begin{cases} 0 & \text{if } T \not\subseteq S \\ \frac{\mu(S)}{\sum_{S' \supseteq T} \mu(S')} & \text{if } T \subseteq S \end{cases}$

  $\uparrow$ up operator



- $D_{k\to\ell}U_{\ell\to k}$: down-up walk $\leftarrow$ implementable for $k-\ell = O(1)$

- Dist on $\Omega_1 \times \cdots \times \Omega_n \leftrightarrow \binom{\Omega_1 \sqcup \cdots \sqcup \Omega_n}{n}$

  down-up $\equiv$ Glauber

- Link of μ at T: $\leftarrow$ dist on $\binom{[n]-T}{k-|T|}$

  μ_T: Dist of $S-T$ when $S\sim\mu$ cond. on $T \subseteq S$.

  "Often" lines belong to same family.

- Local-to-global:

$$\forall \nu: \ D_f(\nu D_{k-|T|\to 1} \,\|\, \mu_T D_{k\to 1}) \leq (1-\rho_T)\, D_f(\nu \,\|\, \mu_T)$$

$$\Downarrow$$

$$\forall \nu: \ D_f(\nu D_{k\to\ell} \,\|\, \mu D_{k\to\ell}) \leq (1-\gamma)\, D_f(\nu \,\|\, \mu)$$

$$\gamma := \min\{\rho_\emptyset, \rho_{\{e_1\}}, \rho_{\{e_1,e_2\}}, \cdots, \rho_{\{e_1,\dots,e_{\ell-1}\}}\}$$

- Useful instatiations:

  * Spectral ind.: $D_f = \chi^2$, $\rho_T = 1 - \frac{c}{k-|T|}$
  * Entropic ind: $D_f = D_{KL}$, $\rho_T = 1 - \frac{c}{k-|T|}$

$$\gamma = \binom{k-\ell}{c} \Big/ \binom{k}{c} \leftarrow \text{int. } c$$

Plan for Today

- Spectral HDX
- Geometry of polynomials

Spectral HDX ($D_f = \chi^2$)

- We want $\lambda_2(U_{1\to k} D_{k\to 1})$ bounds:

$$\underbrace{\frac{1}{k}\begin{bmatrix} \mathbb{P}[1|1] & \mathbb{P}[2|1] & \cdots & \mathbb{P}[n|1] \\ \mathbb{P}[1|2] & \mathbb{P}[2|2] & \cdots & \mathbb{P}[n|2] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}[1|n] & \mathbb{P}[2|n] & \cdots & \mathbb{P}[n|n] \end{bmatrix}}_{\text{bound } \lambda_2(M) \leq O(1)}$$

$\mathbb{P}[j|i] := \mathbb{P}_{S\sim\mu}[j\in S \mid i \in S]$

$D_{ii} = \mathbb{P}[i\in S]$ $\quad X_{ij} = \mathbb{P}[i,j]$

$$M = \underbrace{\mathrm{diag}(\mathbb{E}_{S\sim\mu}[1_S])}_{D}{}^{-1} \cdot \underbrace{\mathbb{E}_{S\sim\mu}[1_S 1_S^t]}_{X}$$

indicator of S

- Symmetric form:

$$D^{-1}X \sim_{\text{similar}} D^{-\frac{1}{2}} X D^{-\frac{1}{2}} = M'$$

- M has real eigs and top eig $= k$.

$$M1 = k1 \quad , \quad M' D^{\frac{1}{2}}1 = k D^{\frac{1}{2}}1$$

$$\|D^{\frac{1}{2}}1\|^2 = \sum_i \mathbb{P}[i] = k$$

- $\lambda_2(M')$ is the same as

$$\lambda_{max}\left(M' - k \cdot \frac{vv^t}{\|v\|^2}\right) \quad \text{for } v = \text{top eig.vec}$$

$$\left(M' = \lambda_1 v_1 v_1^t + \lambda_2 v_2 v_2^t + \cdots + \lambda_n v_n v_n^t\right)$$

- $$\cancel{k} \cdot \frac{vv^t}{\cancel{\|v\|^2}_{\,k}} = \underline{D^{\frac{1}{2}} 1 1^t D^{\frac{1}{2}}}$$

$$M' - k\frac{vv^t}{\|v\|^2} = D^{-\frac{1}{2}}\left(X - \underline{D11^tD}\right) D^{-\frac{1}{2}}$$

$$\underbrace{\mathbb{E}[1_S 1_S^t] - \mathbb{E}[1_S]\mathbb{E}[1_S]^t}_{\mathrm{cov}(\mu)}$$

μ on $\{0,1\}^n$

- So spectral HDX means

$$D^{-\frac{1}{2}} \mathrm{cov}\, D^{-\frac{1}{2}} \preceq O(1) I$$

$$\lambda_{max}\left(D^{-\frac{1}{2}} \mathrm{cov}(\mu) D^{-\frac{1}{2}}\right) = O(1)$$

$$\Updownarrow$$

$$\mathrm{cov}(\mu) \preceq O(1)\, D \qquad \text{diag}(\mathbb{E}[1_S])$$

- Nonsymmetric version:

$$\lambda_{max}\left(D^{-\frac{1}{2}} \mathrm{cov}(\mu) D^{-\frac{1}{2}}\right) = \lambda_{max}\left(D^{-1} \mathrm{cov}(\mu)\right)$$

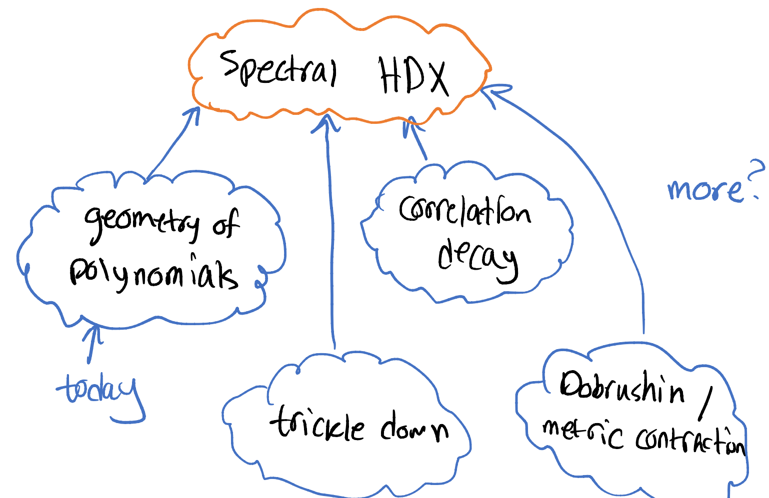
correlation matrix →

$$\begin{bmatrix} \mathbb{P}[1|1]-\mathbb{P}[1] & \cdots & \mathbb{P}[n|1]-\mathbb{P}[n] \\ \vdots & \mathbb{P}[j|i]-\mathbb{P}[i] & \vdots \\ \mathbb{P}[1|n]-\mathbb{P}[1] & \cdots & \mathbb{P}[n|n]-\mathbb{P}[n] \end{bmatrix}$$

So far equiv:

- $\lambda_2(U_{1\to k} D_{k\to 1}) \leq C$
- $\lambda_{max}(\text{correlation}) \leq C$
- $\mathrm{cov}(\mu) \preceq C \cdot \text{diag}(\text{mean}(\mu))$ $\quad \mathbb{E}[1_S]$

How do we establish these?



- Remark: We will come to Entropic HDX and tightening of local-to-global estimates.

HDx via Geometry of Polys

Thm: $\mathrm{cov}(\mu) \succeq \frac{1}{\alpha}\,\mathrm{diag}(\mathrm{mean}(\mu))$

$\Updownarrow$ $\quad \alpha \in [0,1]$

$$\nabla^2 \lg g_\mu(z_1^\alpha, \dots, z_n^\alpha) \succeq 0$$

at $z=1$

$$g_\mu(z) \propto \mathbb{E}_{S\sim\mu}\Big[\prod_{i\in S} z_i\Big]$$

$$z_i \partial_i g_\mu = \mathbb{E}_{S\sim\mu}\Big[\prod z_i \mathbb{1}[i\in S]\Big]$$

Proof: Let $Z = \mathrm{diag}(z_1, \dots, z_n)$

* $Z\nabla g / g = \mathbb{E}[\mathbb{1}_S]$ at $z=1$
* $\dfrac{Z\nabla^2 g\, Z}{g} = \mathbb{E}[\mathbb{1}_S \mathbb{1}_S^t] - \mathrm{diag}(\mathbb{E}[\mathbb{1}_S])$

$\mathbb{P}[i\in S]$ at $z=1$

Let $f = g_\mu(z^\alpha)$.

* $\nabla f = \alpha \cdot Z^{\alpha-1} \cdot \nabla g_\mu(z^\alpha)$
* $\dfrac{Z\nabla f}{f} = \alpha \cdot \mathbb{E}[\mathbb{1}_S]$ at $z=1$
* $\nabla^2 f =$

$$\alpha^2 \cdot Z^{\alpha-1} \nabla^2 g_\mu(z^\alpha) \cdot Z^{\alpha-1} + \alpha(\alpha-1) Z^{\alpha-2}\,\mathrm{diag}(\nabla g_\mu(z^\alpha))$$

* $\dfrac{Z\nabla^2 f\, Z}{f} = \alpha^2\big(\mathbb{E}[\mathbb{1}_S\mathbb{1}_S^t] \cancel{- \mathrm{diag}(\mathbb{E}[\mathbb{1}_S])}\big) - \alpha \cdot \mathrm{diag}(\mathbb{E}[\mathbb{1}_S])$ at $z=1$

* $\left(\dfrac{Z\nabla^2 f\, Z}{f}\right) - \left(\dfrac{Z\nabla f}{f}\right)\left(\dfrac{Z\nabla f}{f}\right)^t =$

$$\alpha^2\Big(\mathrm{cov}(\mu) - \frac{1}{\alpha}\cdot \mathrm{diag}(\mathrm{mean}(\mu))\Big)$$

* So spectral HDx $\iff \dfrac{\nabla^2 f}{f} - \dfrac{\nabla f \cdot \nabla f^t}{f^2} \succeq 0$ at $z=1$

$$\nabla^2 \lg f = \frac{\nabla^2 f}{f} - \frac{\nabla f \cdot \nabla f^t}{f^2}$$

Summary: $\mathrm{cov}(\mu) \preceq C \cdot \mathrm{diag}(\mathrm{mean}(\mu))$ iff

$\lg g_\mu(z_1^{\frac{1}{C}}, \dots, z_n^{\frac{1}{C}})$ concave at $z=1$.

Def: When $\lg g_\mu(z^{\frac{1}{C}})$ concave at all z

we call it C-lg-concave.

↖ also called fractionally lg-concave

[Alimohammadi-A-Shiragur-Vuong]

$C=1$: Simply "lg-concave"

Benefit of all z: Links

$$g_{\mu_T} = \lim_{\substack{z_i \to \infty \\ \text{for } i \in T}} \frac{g_\mu(z)}{\prod_{i\in T} z_i} = \mathbb{E}_S[z^S / z^T]$$

← C-lg-concavity closed under this limit!

Remark: C-lg-concave is the clean case.

For many dists, we do not have all z.

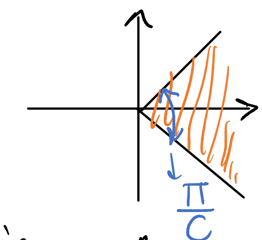
Thm: If g_μ half-plane-stable ← no roots in half-plane [Gårding]

$\Longrightarrow$ lg-concave

Thm: If g_μ has no roots in sector ← sector-stable

$\Longrightarrow$ C-lg-concave

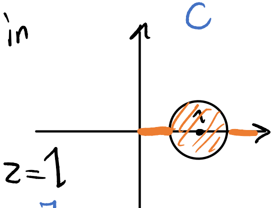
[A-Ivrkov]



Thm: If g_μ has no roots in

$\mathbb{R}_{>0} \cup D(1, \Omega(1))$

$\Longrightarrow$ $O(1)$-lg-concave at $z=1$



[Alimohammadi-A-Shiragur-Vuong]

[Chen-Liu-Vigoda] ← Replaced $\mathbb{R}_{>0}$ with extra assumptions on μ.

Gårding's Theorem

and >0 on $\mathbb{R}^n_{>0}$

Thm: If $p \neq 0$ when $\mathrm{Re}(z_i) > 0 \ \forall i$ and p is hom $\implies$ p is lg-concave!

Toy Case: Suppose p is univariate poly with roots $\in \mathbb{R}_{\leq 0}$. Then $\lg p$ is concave over $\mathbb{R}_{>0}$.

$$\lg p = c + \underbrace{\lg(z+\lambda_1) + \cdots + \lg(z+\lambda_n)}_{\text{each one concave}}$$

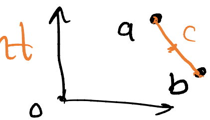
Exercise: univariate with roots $\in \mathbb{R}_{\leq 0}$ $\Longleftrightarrow$ bivariate hom. half-plane-stable

$a_n z^n + \cdots + a_0 \Longleftrightarrow a_n z^n + \cdots + a_0 z'^n$

Note: This is not lg-concavity of coeffs of p, but it is related!

General Case [Gårding]

Take $a, b \in \mathbb{R}^n_{>0}$. Enough to prove lg-concavity on segment



$$q(s,t) := p(sa+tb)$$

only two variables now

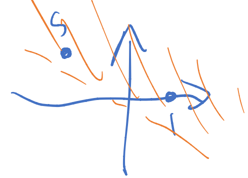
- q is also homogeneous of deg k
- $\mathrm{Re}(s), \mathrm{Re}(t) > 0 \implies q(s,t) \neq 0$.

$$\mathrm{Re}(sa+tb) = \mathrm{Re}(s)a + \mathrm{Re}(t)b$$

Same with every half-plane

$$q = c_0 s^0 t^k + c_1 s^1 t^{k-1} + \cdots + c_k s^k t^0$$

The univariate $q(s,1)$ must have roots in $\mathbb{R}_{\le 0}$. Otherwise $s,1 \in$ half-plane for every $s \notin \mathbb{R}_{\le 0}$!



By factorizing q we get

$$q = (d_1 s + e_1 t) \cdots (d_k s + e_k t)$$

real positive d_i, e_i

Exercise: Check that $\lg(d_i s + e_i t)$ is concave in s, t over $\mathbb{R}^2_{>0}$.

$$\lg q = \sum \lg(d_i s + e_i t)$$

Corollary: The DPP dist defined by $v_1, \dots, v_n \in \mathbb{R}^k$ with

$$\mu(S) \propto \det^2([v_i]_{i \in S})$$

has lg-concave poly $\Rightarrow$ HDX $\Rightarrow$ DU walk has spectral gap $\geq \frac{1}{k}$

Corollary: The spanning tree dist on $\binom{\text{edges}}{|\text{verts}|-1}$ has lg-concave poly $\Rightarrow$ DU walk has spectral gap $\geq \frac{1}{\#\text{verts}-1}$.

$$t_{mix} \leq O(|V| \cdot \lg \#\text{trees}) = O(|V|^2 \lg |V|)$$

Remark: We will show how to get MLSI & tighter mixing time.

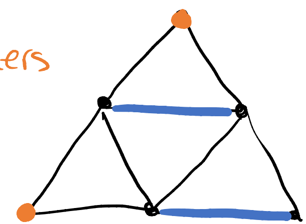
What about other regions $\subseteq \mathbb{C}$?

Example (Monomers)

Take a graph (possibly weighted) and the monomer-dimer system on it.

μ: dist of just monomers

$\mu(S) \propto \#\,\text{PMs on } G-S$

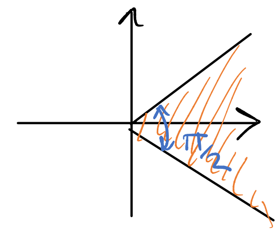


Thm: [Heilmann-Lieb] The monomer dist has $g_\mu(z_1,\dots,z_n) = \mathbb{E}_{S\sim\mu}\left[\prod_{i\in S} z_i\right]$ stable in $\{z \mid \text{Re}(z)>0\}$.

not hom.

no longer equiv to all half planes

Corollary: Hom poly $z_1' \cdots z_n' \, g_\mu\left(\frac{z_1}{z_1'}, \dots, \frac{z_n}{z_n'}\right)$ is stable in sector of aperture $\frac{\pi}{2}$:



$z^S \to z^S z'^{[n]-S}$

This is the generating poly of equiv dist on $\binom{V\times\{0,1\}}{|V|}$.

Corollary: Two-block dynamics on μ has spectral gap $\geq 1/|V|^2$. ← [A-Ivkov]

I will show 4-lg-concave instead of 2.

Application: Sampling/counting monomer-dimer on planar graphs. ← have oracle for μ.)