- **Local-to-global:**

\[ D_f (\nu \| D_{k-1T}) \leq (1 - \rho_T) D_f (\nu \| \mu_T) \]

\[ \rho_T = \min_{i} \rho_{\phi_{i1}} \rho_{\phi_{i2}} \cdots \rho_{\phi_{i,M}} \]

- Useful instantiations:
  * Spectral ind: \( D_f = X^2 \), \( \rho_T = 1 - \frac{c}{k-1T} \)
  * Entropic ind: \( D_f = D_{KL} \), \( \rho_T = 1 - \frac{c}{k-1T} \)

\[ \gamma = \frac{(k-2)}{c} \]

- **Plan for Today**

- Spectral HDX
- Geometry of polynomials

```
Review

- Dist \( \mu \) on \( \binom{[n]}{k} \)

\[ D_{k \rightarrow \ell} (T_1, T_2) = \begin{cases} \frac{1}{c(k)} & \text{if } T_1 \not\subseteq T_2 \\ \frac{1}{c(k)} & \text{if } T_1 \subseteq T_2 \\ 0 & \end{cases} \]

- Dist on \( \Omega_1 \cdots \Omega_n \) \( = \binom{[n]}{k-1T} \)

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- Link of \( \mu \) at \( T \): dist on \( \binom{[n]}{k-1T} \)

\( \mu_T \): Dist of \( S \rightarrow T \) when \( S \approx \mu \) cond. on TCS.

“Often” lines belong to same family.
Spectral HDX ($D_f = \chi^2$)

- We want $\lambda_2(U_{1\rightarrow k}D_{k\rightarrow 1})$ bounds:

$$M = \frac{1}{k} \begin{bmatrix}
I[n \times n] & I[n \times 2] & \cdots & I[n \times 1]
\end{bmatrix}
\begin{bmatrix}
I[2 \times 2] & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
I[n \times n] & I[n \times 1]
\end{bmatrix}$$

$$D_{ij} = \text{ind}(S)$$

$$M = \text{diag}(\mathbb{E}[1_5], \mathbb{E}[1_5]) - \mathbb{E}[1_5 1_5^T]$$

- Symmetric form:

$$D^{-1}X \sim D^{-\frac{1}{2}}XD^{-\frac{1}{2}} = M'$$

- $M$ has real eigs and top eig = $\kappa$.

$$M' = kD_{1/2}11^T = D_{1/2}1^T$$

$$\lambda_{\text{max}}(M' - k \mathbb{V} \mathbb{V}^T)$$ for $\mathbb{V} = \text{top eig vec}$

$$M' = \lambda_1 \mathbb{V}_1 \mathbb{V}_1^T + \lambda_2 \mathbb{V}_2 \mathbb{V}_2^T + \cdots + \lambda_n \mathbb{V}_n \mathbb{V}_n^T$$

$$k = \frac{\mathbb{V} \mathbb{V}^T}{\mathbb{V} \mathbb{V}^T} = D_{1/2}11^T = \lambda_{\text{max}}(M' - D_{1/2}11^T)$$

$$M' - \kappa \mathbb{V} \mathbb{V}^T = D_{1/2}11^T - \lambda \mathbb{V} \mathbb{V}^T$$

$$M' = D_{1/2}11^T - \mathbb{E}[1_5 1_5^T] - \mathbb{E}[1_5] \mathbb{E}[1_5]^T$$

$$\text{COV}(M)$$
- So spectral HDX means

\[ \lambda_{\max}(D^{-1/2} \text{cov}(M) D^{-1/2}) = O(1) \]

\[ \text{cov}(M) \preceq O(1) D \]

- Non-symmetric version:

\[ \lambda_{\max}(D^{-1/2} \text{cov}(M) D^{-1/2}) = \lambda_{\max}(D^{-1} \text{cov}(M)) \]

\[
\begin{bmatrix}
{\text{IP}[11]} & \cdots & \text{IP}[11]
\end{bmatrix}
\begin{bmatrix}
\text{IP}[11] - \text{IP}[j1]
\text{IP}[11] - \text{IP}[j1]
\end{bmatrix}
\]

So far equiv:

- \[ \lambda_2(U_{14} D_{14}) \leq C \]
- \[ \lambda_{\max}(\text{correlation}) \leq C \]
- \[ \text{cov}(M) \preceq C \cdot \text{diag} \text{(mean}(M)) \]

How do we establish these?

Spectral HDX

- Remark: We will come to Entropic HDX and tightening of local-to-global estimates.
\[ \text{Thm: } \text{cov}(M) = \frac{1}{\alpha} \text{diag(mean}(\mu)) \]

\[ \nabla^2 \log g_\mu(z_1, \ldots, z_n) \]

\[ \alpha \in [0, 1] \]

\[ g_\mu(z) \propto \text{E} \left[ \prod_{i=1}^n z_i \right] \]

\[ \text{Z diag} \mu = \text{E} \left[ \prod_{i=1}^n z_i \right] \text{E}[1 \text{ if } z_i > 0] \]

**Proof:** Let \( Z = \text{diag}(z_1, \ldots, z_n) \)

\[ \nabla^2 g \Big|_{z=1} = 15 \text{ if } 1 = 1 \]

\[ \frac{Z \nabla^2 g}{g} = 15 \text{ if } 1 = 1 \]

\[ \frac{Z \nabla^2 g}{g} \]

\[ \nabla^2 \log f = \frac{\nabla^2 f}{f} - \frac{\nabla f \cdot \nabla f^t}{f \nabla f} \]

\[ \text{at } z = 1 \]

\[ \text{Let } f = g_\mu(z^\alpha) \]

\[ \nabla f = \alpha \cdot z^{\alpha-1} \cdot \nabla g_\mu(z^\alpha) \]

\[ \frac{Z \nabla f}{f} = \alpha \cdot E[1_{z>1}] \text{ at } z = 1 \]

\[ \nabla^2 f = \alpha^2 \cdot z^{2 \alpha-2} \cdot \nabla g_\mu(z^\alpha) + \alpha(\alpha-1) \cdot z^{\alpha-2} \cdot \text{diag}(\nabla g_\mu(z^\alpha)) \]

\[ \frac{Z \nabla^2 f}{f} = \alpha^2 \cdot (E[1_{z>1}] - \text{diag}(15[1])) \text{ at } z = 1 \]

\[ (Z \nabla^2 f)^t = (Z \nabla f)(Z \nabla f)^t = \alpha^2 (\text{cov}(\mu) - \frac{1}{\alpha} \cdot \text{diag(mean}(\mu))) \]

\[ \text{So Spectral HDX } \iff \frac{\nabla^2 f}{f} = \frac{\nabla f \cdot \nabla f^t}{f^2} \]

\[ \text{at } z = 1 \]
Summary: \( \text{Cov}(\mu) \propto \text{C. diag} \{\text{mean} (\mu)\} \) iff 
\[ \lg g_\mu(z_1, \ldots, z_n) \] concave at \( z = 1 \).

Remark: \( C \)-lg-concave is the clean case.
For many dists, we do not have all \( z \).

Def: When \( \lg g_\mu(z_1, \ldots, z_n) \) is concave at all \( z \), we call it \( C \)-lg-concave.

\[ \text{also called} \quad \text{fractionally lg-concave} \]

\( C = 1 \): Simply "lg-concave"

Benefit of all \( z \): Links

\[ g_\mu = \lim_{M \to \infty} g_\mu(z) \quad \text{C-lg-concavity} \]

Closed under this limit!

\[ = \left( \sum_{T} z T^T \right) \]

Thm: If \( g_\mu \) half-plane-stable \( \Rightarrow \) lg-concave

\[ \Rightarrow \text{sector-stable} \]

Thm: If \( g_\mu \) has no roots in sector \( \Rightarrow C \)-lg-concave

Thm: If \( g_\mu \) has no roots in \( \mathbb{R}^+ \cup \mathbb{D}(1, \infty(1)) \)

\[ \Rightarrow \mathcal{O}(1) \]-lg-concave at \( z = 1 \)

\[ \text{[Alimohammadi-A-Shirgar-Unrong]} \]

\[ \text{[Chen-Liu-Vigoda]} \quad \text{Replaced} \ |R|_\infty \text{ with \ extra \ assumptions \ on} \ \mu. \]
Garding's Theorem and $\mathbb{R}^n_{>0}$

Thm: If $p \not< 0$ when $\text{Re}(z_i) > 0 \forall i$ and $p$ is hom $\Rightarrow p$ is lg-concave!

Toy Case: Suppose $p$ is univariate poly with roots $\in \mathbb{R}_{\leq 0}$. Then $\lg p$ is concave over $\mathbb{R}_{>0}$.

$\lg p = c + \lg(z_1^\lambda_1) + \cdots + \lg(z_n^\lambda_n)$

each one concave

Exercise: univariate with roots $\in \mathbb{R}_{\leq 0} \iff$

$\ a_n z^n + \cdots + a_0 \iff a_n z^n + \cdots + a_0 z^k$

bivariate hom. half-plane-stable

Note: This is not lg-concavity of coeffs of $p$, but it is related!

General Case [Garding]

Take $a, b \in \mathbb{R}^n_{>0}$. Enough to prove lg-concavity on segment $\mathbb{R}^n_{>0}$.

$q(s,t) := p(sa + tb)$

only two variables now

$q$ is also homogeneous of deg $k$

$\text{Re}(s), \text{Re}(t) > 0 \Rightarrow q(s,t) > 0$.

$\text{Re}(sa + tb) = \text{Re}(s)a + \text{Re}(t)b$

same with every half-plane

$q = c_0 s^k + c_1 s^{k-1}t + \cdots + c_k t^k$
The univariate \( q(\lambda) \) must have roots in \( \mathbb{R}^2_{\geq 0} \). Otherwise, it lies in the half-plane for every \( \lambda \in \mathbb{R} \).

By factorizing \( q \) we get

\[
q = (d_1 + e_1^t) \cdots (d_k + e_k^t)
\]

real positive \( d_i, e_i \)

**Exercise:** Check that \( \log (d_i + e_i^t) \) is concave in \( \text{S} \) over \( \mathbb{R}^2_{\geq 0} \).

\[
\log q = \sum \log (d_i + e_i^t)
\]

**Corollary:** The PPP dist defined by \( v_1, \ldots, v_n \in \mathbb{R}^k \) with

\[
\mu(s) \propto \exp^2([v_i]_{i < s})
\]

has \( \log \)-concave poly \( \Rightarrow \) HDX \( \Rightarrow \)

DU walk has spectral gap \( \geq \frac{1}{k} \)

**Corollary:** The spanning tree dist on

\[
(\text{edges}) \text{ has } \log \text{-concave poly} \Rightarrow
\]

DU walk has spectral gap \( \geq \frac{1}{\#VeAs - 1} \)

\[
t_{\text{mix}} \leq O(\|v\| \cdot \log \#\text{strees}) = O(\|v\|^2 \log \|v\|)
\]

**Remark:** We will show how to get MLSF & tighter mixing time.
What about other regions $\mathcal{C}$?

**Example (Monomers)**

Take a graph (possibly weighted) and the monomer-dimer system on it.

$\mu$: dist of just monomers

$\mu(s) \propto \# \text{PMs on } G-S$

**Thm. [Heilmann-Lieb]** The monomer dist has $g_{\mu}(z_1, -z_n) = \sum_{S \in \mathcal{S}} \left( \prod_{i \in S} z_i \right)$ in $\{z \mid \text{Re}(z) > 0\}$. not hom.

This is the generating poly of equiv dist on $\left( \mathbb{V} \times \mathbb{P}^{\infty}, 1/1\right)$

**Corollary:** Two-block dynamics on $\mu$ has spectral gap $\geq 1/|\mathbb{V}|^2$. $\left[ \text{A-Ivelev} \right]$

I will show 4-lg-concave instead of 2.

**Application:** Sampling/counting monomer-dimer on planar graphs. $\left[ \text{have oracle for } \mu \right]$