Review

- Patel-Regts trick:

  Coeffs of matching poly

  $\Updownarrow$ polynomial maps

  moments $\sum \lambda_i^{-\alpha}$ of roots of matching poly

  additive: $f(G_1+G_2) = f(G_1)+f(G_2)$

  Additive funcs written as $\sum c_i \operatorname{ind}(H_i, \cdot)$

  can be computed in $\operatorname{poly}(n) \cdot \Delta^{O(k)}$ time

  where $k = \max\{|H_i|\}$. ← assuming we know how to "brute-force" f

- Corollary:

  #matchings has FPTAS for $\Delta = O(1)$

  Open: Remove $\Delta = O(1)$. Jerrum-Sinclair's Markov chain doesn't need it!

- Matching real-rooted ⟶ $\log$-concave coeffs

"similar"

- HDX view: dist μ on $\binom{[n]}{k}$

- Noise operators: $D_{k\to \ell} \in \mathbb{R}_{\geq 0}^{\binom{[n]}{k} \times \binom{[n]}{\ell}}$ sends S to unif. random $T \in \binom{S}{\ell}$

$$D_{k\to\ell}(S,T) = \begin{cases} 0 & \text{if } T \not\subseteq S \\ \frac{1}{\binom{k}{\ell}} & \text{if } T \subseteq S \end{cases}$$

$U_{\ell\to k} = D^{\circ}_{k\to\ell}$ is the time-reversal (w.r.t μ)

$D_{k\to\ell} U_{\ell\to k}$ is the $k \leftrightarrow \ell$ down-up walk

Algorithmically relevant only for $\ell = k - O(1)$

Analytically relevant for all ℓ!

- Informal Def of HDX: $D_{k\to 1}$ contracts χ^2.

- Generating poly: $g_\mu(z_1, \dots, z_n) = \sum_S \mu(S) z^S$ ← monomial

- Det dists: $g_\mu = \det(z_1 A_1 + \dots + z_n A_n)$ ← A_i rank 1 PSD.

  Example: Spanning trees with

  A_i = Laplacian of edge i ← drop one row/col

Thm: When μ is determinantal

$$g_\mu(z_1, \dots, z_n) \neq 0 \quad \text{if} \quad \mathrm{Re}(z_i) > 0 \ \forall i$$

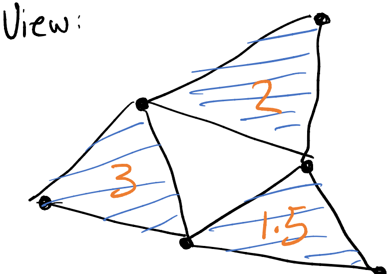
"half-plane stability" any half-plane equiv.

Plan for today:

- Formal def of HDX
- Local-to-global: from $D_{k \to 1}$ to $D_{k \to \ell}$
- Show det. dists are HDX ($g_\mu \leftrightarrow$ HDX)

High-Dimensional Expanders

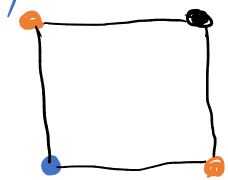
- Setup: $\mu : \underbrace{\binom{[n]}{k}}_{\text{Subsets of size } k \text{ out of } \{1, \dots, n\}} \longrightarrow \mathbb{R}_{\geq 0}$
- For $k = 2$, these are weighted graphs.
- Hypergraph View:



Example: (Spanning trees): $\mu : \binom{\text{edges}}{\text{verts} - 1} \longrightarrow \mathbb{R}_{\geq 0}$

- Many discrete distributions can be viewed this way. (even on prod spaces)

Example (Graph Coloring)



- $\mu : \{\text{colorings}\} \to \mathbb{R}_{\geq 0}$
- View colorings as subsets of size n from

$$[n] \times [q] = \{(1,1), \ldots, (1,q), \ldots, (n,q)\}$$

- Assign zero weight to invalid subsets.

HDX: $D_{k \to 1}$ contracts χ^2 and same for conditionings of μ

↑ ignore for now

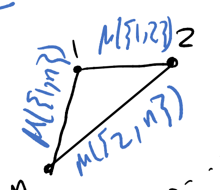
This contraction is equiv. to

$$\lambda_2(D_{k\to 1} U_{1 \to k}) = \lambda_2(\underbrace{U_{1\to k} D_{k \to 1}}_{n \times n})$$

Example. ($k=2$, graphs)

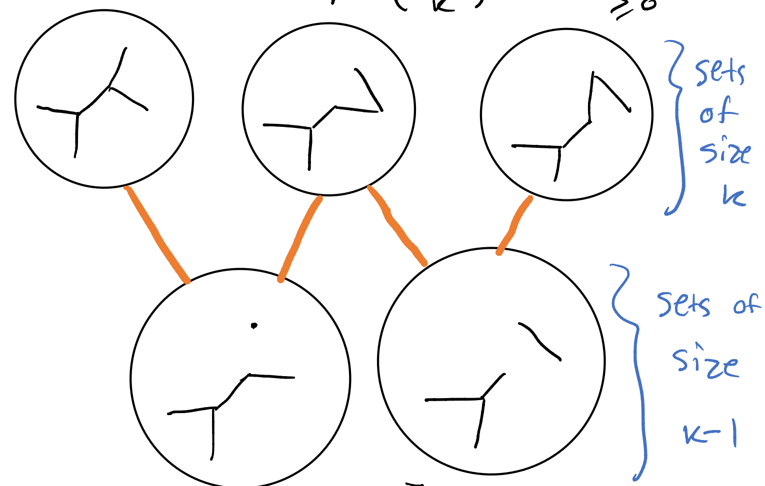
$U_{1\to 2} D_{2 \to 1}$ is lazy random walk!

Stay w.p. $\frac{1}{2}$



Down-Up Random Walks

To sample from $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$,



- Start from $S \in \binom{[n]}{k}$,
- Drop u.a.r. elem i to get $S - i$
- Add elem j w.p. $\propto \mu(S - i + j)$
- Repeat

Generalization: Drop $k-\ell$ and add $k - \ell$

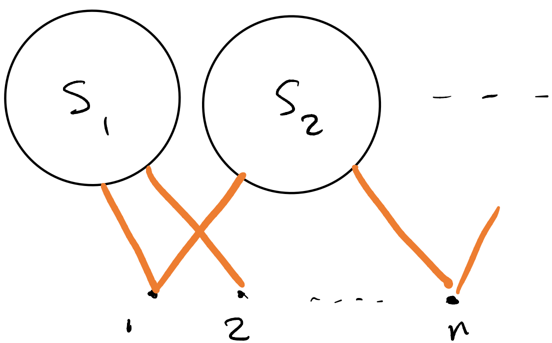
$$D_{k \to \ell} U_{\ell \to k}$$

$k \leftrightarrow 1$ Walk

Even though $k-\ell = O(1)$ is the algorithmically interesting case, we study the $k \leftrightarrow 1$ walk for analysis.

Thm (Local-to-Global) [Kaufman-Oppenheim, Alev-Lau, ...]

If $D_{k\to 1}$ contracts f-div for μ and links/conditionings of μ so does $D_{k\to \ell}$



$k \leftrightarrow 1$ walk

HDX framework for MC analysis:

① Somehow show ← today based on half-plane stability

$$D_f(\nu D_{k\to 1} \| \mu D_{k\to 1}) \leq \frac{C}{k} D_f(\nu \| \mu)$$

Standard is χ^2 — $C=1$ for today

② Conclude same for conds of μ — usually automatic by self-reducibility

③ By local-to-global get

$$D_f(\nu D_{k\to \ell} \| \mu D_{k\to \ell}) \leq \left(1 - \frac{\binom{k-\ell}{C}}{\binom{k}{C}}\right) D_f(\nu \| \mu)$$

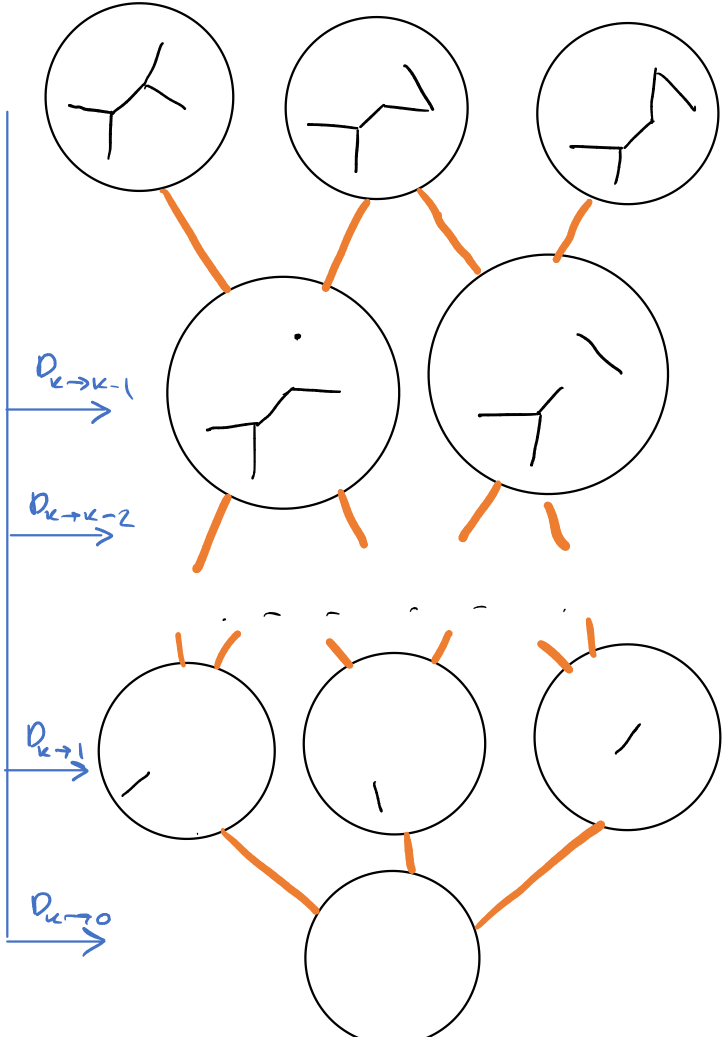
assuming C int. similar for $C \notin \mathbb{Z}$

Remark: Need $k-\ell \geq C$.

In that case contraction $\simeq 1 - 1/k^C$

$\Rightarrow$ relaxation time $= O(k^C)$ — good when $C = O(1)$.

Remark: Step 1 is difficult!

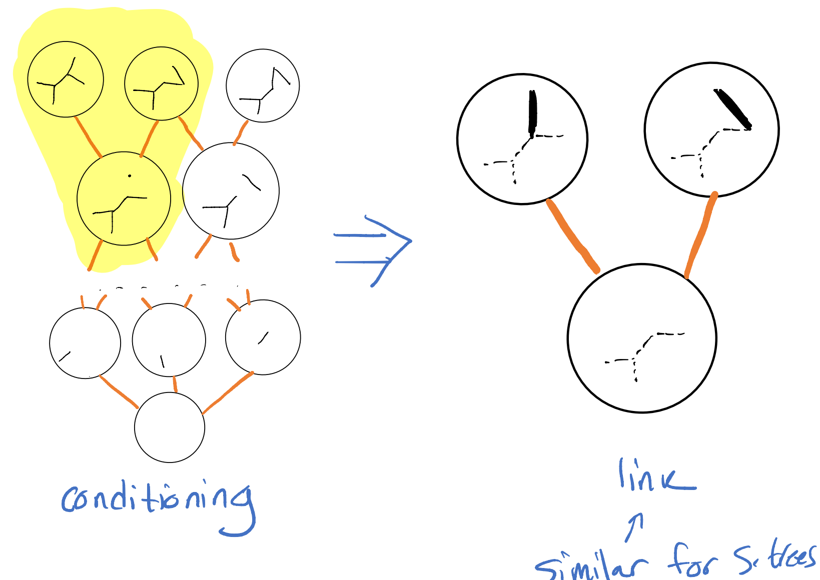


Conditionings/links: Given T of size $\leq k$

$\mu|T$ is dist of $S\sim\mu$ cond. on $T\subseteq S$.

We call dist of $S-T$ link of μ at T

and denote it by $\mu_T : \binom{[n]-T}{k-|T|} \to \mathbb{R}_{\geq 0}$



Thm (Local-to-Global)

Suppose $D_{(k-|T|)\to 1}$ has $1-\rho_T$ contraction of f-div w.r.t. μ_T. Then

$$D_f(\nu D_{k\to\ell} \,\|\, \mu D_{k\to\ell}) \leq (1-\gamma)\, D_f(\nu \,\|\, \mu)$$

$$\gamma := \min\{\rho_\emptyset \, \rho_{\{e_1\}} \cdots \rho_{\{e_1,\dots,e_{\ell-1}\}}\}$$

Corollary: Suppose $\rho_T = 1 - \frac{C}{k-|T|}$. Then

$$\gamma = \left(1-\frac{C}{k}\right)\left(1-\frac{C}{k-1}\right)\cdots\left(1-\frac{C}{k-\ell+1}\right)$$

$$= \frac{\cancel{k-C}}{k}\cdot\frac{\cancel{k-C-1}}{k-1}\cancel{\cdots}\frac{k-\ell+1-C}{\cancel{k-\ell+1}} = \frac{\binom{k-\ell}{C}}{\binom{k}{C}}$$

Corollary: For $C=1$ this is $(k-\ell)/k$

Example (uniform)

- μ unif on $\binom{[n]}{k} \Rightarrow \mu_T$ unif on $\binom{[n]-T}{k-|T|}$

  similar

- We have $(U_{1\to k} D_{k\to 1})_{ij} = \frac{1}{k}\mathbb{P}[j\in S \mid i \in S]$

$$\frac{1}{k}\begin{bmatrix} 1 & \frac{k-1}{n-1} & \cdots & \frac{k-1}{n-1} \\ \frac{k-1}{n-1} & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{k-1}{n-1} & \cdots & \cdots & 1 \end{bmatrix} =$$

$$\frac{1}{k}\left(\frac{k-1}{n-1}\overset{\text{all-ones}}{J} + \left(1-\frac{k-1}{n-1}\right)\overset{\text{identity}}{I}\right)$$

$$\Rightarrow \lambda_2 \leq \frac{1-\frac{k-1}{n-1}}{k} \leq \frac{1}{k} \leftarrow C=1$$

- $D_{k\to\ell}$ contracts χ^2 divergence by ℓ/k.

Proof of Local-to-Global:

- Fix f-div.

$$D_f(\nu \| \mu) = \mathbb{E}_{S\sim\mu}\left[f\left(\frac{\nu(S)}{\mu(S)}\right)\right] - f\left(\mathbb{E}\left[\frac{\nu(S)}{\mu(S)}\right]\right)$$

- For a set T of size t define (abuse of notation)

$$\nu(T) = \nu D_{k\to t}(T), \quad \mu(T) = \mu D_{k\to t}(T)$$

- Then

$$D_f(\nu \| \mu) = \mathbb{E}_{S\sim\mu}\left[f\left(\frac{\nu(S)}{\mu(S)}\right)\right] - f\left(\frac{\nu(\emptyset)}{\mu(\emptyset)}\right)$$

- Key: Think of sampling $S\sim\mu$ and unif randomly permuting its elements to get $X_1, \dots, X_k$.

- Then $\underbrace{\{X_1, \dots, X_t\}}_{\text{let's call } S_t} \sim \mu D_{k\to t}$

$$f\left(\frac{\nu(S_t)}{\mu(S_t)}\right)$$

- We have

$$D_f(\nu\|\mu) = \mathbb{E}\left[f\left(\frac{\nu(S_k)}{\mu(S_k)}\right) - f\left(\frac{\nu(\overset{\emptyset}{\cancel{S_0}})}{\mu(\underset{\emptyset}{\cancel{S_0}})}\right)\right]$$

- What about $D_f(\nu D_{k\to\ell} \| \mu D_{k\to\ell})$?

$$\mathbb{E}\left[f\left(\frac{\nu(S_\ell)}{\mu(S_\ell)}\right) - f\left(\frac{\nu(\emptyset)}{\mu(\emptyset)}\right)\right]$$

- Let $Z_t = f\left(\frac{\nu(S_t)}{\mu(S_t)}\right)$

$$D_f(\nu\|\mu) = \mathbb{E}[Z_k - Z_0]$$

$$D_f(\nu D_{k\to\ell} \| \mu D_{k\to\ell}) = \mathbb{E}[Z_\ell - Z_0]$$

want to show small

- We know $\mathbb{E}[Z_1 - Z_0] \leq (1-\rho_\emptyset)\cdot\mathbb{E}[Z_k - Z_0]$

Claim: We also have (from links):

$$\mathbb{E}[Z_{t+1} - Z_t \mid S_t] \leq (1-\rho_{S_t})\,\mathbb{E}[Z_k - Z_t \mid S_t]$$

Proof of Claim:

- Dist of $X_{t+1}, \ldots, X_k$ conditioned on S_t is the same permutation process applied to μ_{S_t} ← link.

- $\frac{\nu(T)}{\mu(T)}$ for $T \geq S_t$ is the same as

$$\frac{\nu_{S_t}(T-S_t)}{\mu_{S_t}(T-S_t)}$$

$\square$

We know that

$$\mathbb{E}[Z_{t+1} - Z_t | S_t] \leq (1-\rho_{S_t}) \mathbb{E}[Z_k - Z_t | S_t]$$

$$\Downarrow$$

$$\mathbb{E}[Z_k - Z_{t+1} | S_t] \geq \rho_{S_t} \cdot \mathbb{E}[Z_k - Z_t | S_t]$$

This means

$$Y_t := \frac{Z_k - Z_t}{\rho_{S_0} \rho_{S_1} \cdots \rho_{S_{t-1}}}$$

is a submartingale:

$$\mathbb{E}[Y_{t+1} | S_t] \geq \mathbb{E}[Y_t | S_t]$$

So $\mathbb{E}[Y_\ell] \geq \mathbb{E}[Y_0] = D_f(\nu \| \mu)$

and

$$\mathbb{E}[Y_\ell] \leq \frac{\mathbb{E}[Z_k - Z_\ell]}{\min\{\rho_\phi \rho_{\{e_1\}} \cdots \rho_{\{e_1, \ldots, e_\ell\}}\}} \leftarrow \gamma$$

$$\Rightarrow \mathbb{E}[Z_k - Z_\ell] \geq \gamma \cdot \mathbb{E}[Z_k - Z_0]$$

$$\Rightarrow \underbrace{\mathbb{E}[Z_\ell - Z_0]}_{= D_f(\nu D_{k \to \ell} \| \mu D_{k \to \ell})} \leq (1-\gamma) \underbrace{\mathbb{E}[Z_k - Z_0]}_{= D_f(\nu \| \mu)}$$

$\square$

So far:

μ and links have D_f-contracting $D_{\bullet\to 1}$

$$\Downarrow$$

μ has D_f-contracting $D_{k\to \ell}$

Useful specialization:

$D_{\bullet\to 1}$ contracts by $\frac{C}{k}$

$\Rightarrow D_{k\to \ell}$ contracts by $1-\frac{\binom{k-\ell}{C}}{\binom{k}{C}}$

- Spectral Independence: This for χ^2 and $C=O(1)$
- Entropic Independence: This for D_{KL} and $C=O(1)$

$\Rightarrow$ Poly-time sampling via $D_{k\to k-O(1)}$

How to prove $\boxed{D_{\bullet\to 1} \text{ contracts}}$?

This is the difficult part.

We will see many techniques for it.

today: half-plane stability

- In the χ^2 case we want to bound $\lambda_2(U_{1\to k}D_{k\to 1})$:

$$\underbrace{\begin{array}{cc} & \downarrow j \\ i\rightarrow & \left(\begin{array}{c} \\ \dfrac{\mathbb{P}[j\in S \mid i\in S]}{k} \\ \\ \end{array}\right)\end{array}}_{\text{the matrix we need to analyze}}$$

convenient

- Let $\varphi_{ij} := \mathbb{P}[j \mid i]$.

1 on diagonal.

Claim: $\lambda_2(\varphi) \le C$ if and only if

$\lg g_\mu(z_1^{\frac{1}{C}}, \ldots, z_n^{\frac{1}{C}})$ is concave at $z=1$.

$$\Updownarrow$$

$$\left(\nabla^2 \lg g_\mu(z_1^{\frac{1}{C}}, \ldots, z_n^{\frac{1}{C}})\right)\Big|_{z=1} \preccurlyeq 0$$

Proof: Homework calculations. :) $\square$

We will prove g_μ is half-plane-stable

$\Rightarrow$ $\lg g_\mu$ is concave over $\mathbb{R}^n_{>0}$.

[Gårding]

Lg-concave polynomial: $\lg g_\mu$ concave over $\mathbb{R}^n_{>0}$

another name: Lorentzian

Toy Case: Suppose p is univariate poly with roots $\in \mathbb{R}_{<0}$. Then $\lg p$ is concave over $\mathbb{R}_{>0}$.

$$\lg p = C + \lg(z+\lambda_1) + \cdots + \lg(z+\lambda_n)$$

each one concave

Note: This is not lg-concavity of coeffs of p, but it is related!