Review

- Patel-Regts trick:
  Coeffs of matching poly
  \( \sum \lambda_i^{-\alpha} \) of roots of matching poly
  additive: \( f(G_1 \cup G_2) = f(G_1) + f(G_2) \)

Additive funs written as \( \sum C_i \text{ind}(H_i) \cdot \)

- Patel-Regts trick:
  Coeffs of matching poly
  \( \sum \lambda_i^{-\alpha} \) of roots of matching poly
  additive: \( f(G_1 \cup G_2) = f(G_1) + f(G_2) \)

Additive funs written as \( \sum C_i \text{ind}(H_i) \cdot \)

can be computed in \( \text{poly}(n), \Delta \) time

where \( \kappa = \max \|H\|_p \). Assuming we know how to "brute-force" \( f \)

- Corollary:

  \#matchings has FPTAS for \( \Delta = O(1) \)

- Open: Remove \( \Delta = O(1) \) - Jerrum-Sinclair's
  Markov chain doesn't need it!

- Matching real-rooted \( \rightarrow \) \( \kappa \)-concave coeffs

- HDX view: dist \( \mu \) on \( \binom{[n]}{\kappa} \)

- Noise operators:
  \( D_{l \to k} \in R^{\binom{[n]}{\kappa}} \times \binom{[n]}{\ell} \)

  \( D_{l \to k}(S, T) = \begin{cases} 0 & \text{if } T \neq S \\ \frac{1}{\binom{\ell}{k}} & \text{if } T = S \end{cases} \)

  Sends \( S \) to uniform random \( T \in \binom{S}{\ell} \)

  \( U_{l \to k} = D_{k \to l}^* \) is the time-reversal (w.r.t \( \mu \))

  \( D_{k \to l} U_{l \to k} = \) the \( k \to l \) down-up walk

  Algorithmically relevant only for \( l = k - O(1) \)

  Analytically relevant for all \( l \)!

- Informal Def of HDX: \( D_{k \to l} \) contracts \( \chi_2 \)

- Generating poly:
  \( g_{\mu}[z_1, \ldots, z_n] = \sum_{S} \mu(S) z^S \)

- Det dists:
  \( g_{\mu} = \det(z_1 A_1, \ldots, z_n A_n) \)

  Example: Spanning trees with

  \( A_i = \text{Laplacian of edge } i \)

\( \approx \) deep one row/col
Thm: When \( \mu \) is determinantal

\[
g_\mu(z_1, \ldots, z_n) \neq 0 \text{ if } \Re(z_i) > 0 \quad \forall i
\]

“half-plane stability” any half-plane equiv.

High-Dimensional Expanders

Setup: \( \mu: ([n]) \rightarrow (\mathbb{R}_{\geq 0}) \)

Subsets of size \( k \) out of \([n] - 1,n\)?

- For \( k=2 \), these are weighted graphs.
- Hypergraph View:

Example: (Spanning trees): \( \mu: (\text{edges}) \rightarrow \mathbb{R}_{\geq 0} \)

- Many discrete distributions can be viewed this way. (Even on prod spaces)

Plan for today:
- Formal def of HDX
- Local-to-global: from \( D_{k-1} \) to \( D_{k}\)
- Show det. dists are HDX (\( g_\mu \leftrightarrow \text{HDX} \))
Example (Graph Coloring)
- $\mu: \{\text{colorings}\} \rightarrow \mathbb{R}_{\geq 0}$
- View colorings as subsets of size $n$ from $[n] \times [q] = \{(1,1), \ldots, (n,q)\}$
- Assign zero weight to invalid subsets.

HDX: $D_k \rightarrow 1$

This contraction is equiv. to $\lambda_2(D_k \rightarrow 1 U_{1 \rightarrow k}) = \lambda_2(U_{1 \rightarrow k} D_k \rightarrow 1)$. Ignore for now.

Example. ($k=2$, graphs)
$U_{1 \rightarrow 2} D_2 \rightarrow 1$ is lazy random walk!

$\text{Down-Up Random Walues}$

To sample from $\mu_s(\mathbb{E}_k) \rightarrow (\mathbb{R}_{\geq 0})$

- Start from $S \in ([n]_k)$,
- Drop u.a.r. elem $i$ to get $S-i$
- Add elem $j$ w.p. $\alpha(M(S-i+j]$
- Repeat

Generalization: Drop $k-l$ and add $k-2$

$D_{k-l} U_{l \rightarrow k}$
**k \leftarrow 1 \text{ Walk}**

Even though $k-l=O(1)$ is the algorithmically interesting case, we study the $k \leftarrow 1$ walk for analysis.

**Thm (Local-to-Global)** [Kaufman-Oppenheim, Alev-Law, ...]

If $D_{k\rightarrow 1}$ contracts f-div for $\mu$ and links/conditionings of $\mu$ so does $D_{k\rightarrow \ell}$.

**HDX framework for MC analysis:**

1. Somehow show $\cdots$ today based on half-plane stability

\[ D_f(\nu D_{k\rightarrow 1} \| \mu D_{k\rightarrow 1}) \leq \frac{C}{k} D_f(\nu \| \mu) \]

Standard is $X^2$

2. Conclude same for $\cdots$ of $\mu$

3. By local-to-global get

\[ D_f(\nu D_{k\rightarrow \ell} \| \mu D_{k\rightarrow \ell}) \leq \left(1 - \frac{(k-1)}{C}\right) D_f(\nu \| \mu) \]

assuming $C$ int. similar for $C\ell$

**Remark:** Need $k-l \geq C$.

In that case contraction $\cdots$ $1-\frac{1}{\nu}\ell$

$\Rightarrow$ relaxation time $= O(C)$

**Remark:** Step 1 is difficult! good when $C=O(1)$. 

\[ \cdots \]
Conditionings/links: Given $T$ of size $\leq k$, $\mu|T$ is dist of Sum cond. on TCS. We call dist of $S-T$ line of $\mu$ at $T$ and denote it by $\mu_T: (\mathbb{N}^3-T)_{k-1T}\rightarrow \mathbb{R}_{\geq 0}$. Similar for $S$-trees.
Thm (Local-to-Global)
Suppose $D_{k-1T} \rightarrow 1$ has $1 - \rho_T$ contraction of $f$-div w.r.t. $\mu_T$. Then

$$D_f (\nu D_{k-1} || \mu D_{k-1}) \leq (1-\delta) D_f (\nu || \mu)$$

$$\delta := \min \left\{ \rho_{\phi} \rho_{e_i} \cdots \rho_{e_i - e_{i-1}} \right\}$$

**Corollary:**
Suppose $\rho_T = 1 - \frac{C}{k-1T}$. Then

$$\delta = (1 - \frac{C}{k}) (1 - \frac{C}{k-1}) \cdots \left(1 - \frac{C}{k-l+1}\right)$$

$$= \frac{k-C}{k} \cdot \frac{k-C-1}{k-1} \cdots \frac{k-l-C}{k-l+1} = \frac{(k-l)C}{(k-C)}$$

**Corollary:** For $C=1$ this is $(k-l)k$.

Example (Uniform)
- $\mu$ unif on $[\gamma]_k \Rightarrow \mu_T$ unif on $[\gamma]_{k-1T}$
- We have $V_1 \rightarrow k D_k \rightarrow 1 = \frac{1}{k} \mathbb{P}_{j \in S | i \in S}$

$$\frac{1}{k} \mathbb{I} = \frac{1}{k} \left( \frac{k-1}{n-1} I + \left(1 - \frac{k-1}{n-1}\right) I \right)$$

$$\Rightarrow \lambda_2 \leq \frac{1 - \frac{k-1}{n-1}}{k} \leq \frac{1}{k}$$

- $D_{k-1}$ contracts $\chi^2$ divergence by $\frac{1}{k}$. 
Proof of Local-to-Global:

- Fix \( f \)-div.

\[ D_f(\nu \| \mu) = \left[ \mathbb{E}_{s \sim \mu} [f(\frac{\nu(s)}{\mu(s)})] - f(\mathbb{E}[\frac{\nu(s)}{\mu(s)}]) \right] \]

- For a set \( T \) of size \( t \) define

\[ \gamma(T) = \nu D_{k \rightarrow t}(T), \quad \mu(T) = \mu D_{k \rightarrow t}(T) \]

- Then

\[ D_f(\nu \| \mu) = \left[ \mathbb{E}_{s \sim \mu} [f(\frac{\nu(s)}{\mu(s)})] - f(\frac{\nu(\emptyset)}{\mu(\emptyset)}) \right] \]

- Key: Think of sampling \( s \sim \mu \) and unit randomly permuting its elements to get \( X_1, \ldots, X_k \).

- Then \( \{X_1, \ldots, X_t\} \sim \mu D_{k \rightarrow t} \)

\[ f(\frac{\nu(s_t)}{\mu(s_t)}) \]

- We have

\[ D_f(\nu \| \mu) = \mathbb{E}\left[f\left(\frac{\nu(\emptyset)}{\mu(\emptyset)}\right) - f\left(\frac{\nu(\emptyset)}{\mu(\emptyset)}\right)\right] \]

- What about \( D_f(\nu D_{k \rightarrow t} \| \mu D_{k \rightarrow t}) \)?

\[ \mathbb{E}\left[f\left(\frac{\nu(\emptyset)}{\mu(\emptyset)}\right) - f\left(\frac{\nu(\emptyset)}{\mu(\emptyset)}\right)\right] \]

- Let \( Z_t = f\left(\frac{\nu(s_t)}{\mu(s_t)}\right) \)

\[ D_f(\nu \| \mu) = \mathbb{E}[Z_k - Z_0] \]

\[ D_f(\nu D_{k \rightarrow t} \| \mu D_{k \rightarrow t}) = \mathbb{E}[Z_k - Z_0] \]

We want to show small

- We know \( \mathbb{E}[Z_k - Z_0] \leq (1 - p) \cdot \mathbb{E}[Z_k - Z_0] \)

Claim: We also have (from links):

\[ \mathbb{E}[Z_{t+1} - Z_t | s_t] \leq (1 - p_{s_t}) \mathbb{E}[Z_k - Z_t | s_t] \]
Proof of Claim: conditioned on $S_t$

- $\text{Dist of } X_{t+1}, \cdots, X_k \text{ is the same}$
  
  permutation process applied to $\nu_{S_t}$

- $\frac{\nu(T)}{\mu(T)} \text{ for } T \geq S_t \text{ is the same as}$
  
  $\begin{align*}
  \frac{\nu'_S(T-S_t)}{\mu'_S(T-S_t)}
  \end{align*}$

This means

$\begin{align*}
Y_t := \frac{Z_{k-2}}{\prod_{i=0}^{S_t-1} \rho_i}
\end{align*}$

is a submartingale:

$\begin{align*}
\mathbb{E}[Y_{t+1} | S_t] \geq \mathbb{E}[Y_t | S_t]
\end{align*}$

So $\mathbb{E}[Y_t] \geq \mathbb{E}[Y_0] = D_f(\nu | \mu)$

and

$\begin{align*}
\mathbb{E}[Y_t] \leq \frac{\left[ \mathbb{E}[Z_{k-2} - Z_0] \right]}{\min \{ \rho_0, \rho_1, \cdots, \rho_{S_t-1} \}} \leq \delta
\end{align*}$

$\Rightarrow \mathbb{E}[Z_{k-2}] \geq \delta \cdot \mathbb{E}[Z_{k-2}]$

$\Rightarrow \mathbb{E}[Z_{k-2}] \leq (1 - \delta) \mathbb{E}[Z_{k-2}]$

$\Rightarrow D_f(\nu_{Z_{k-2}} \parallel \mu_{Z_{k-2}}) \leq D_f(\nu | \mu)$

\[ \square \]
So far:

μ and links have $D_f$-contracting $D \rightarrow 1$

$\downarrow$

μ has $D_f$-contracting $D_{k \rightarrow e}$

Useful specialization:

$D \rightarrow 1$ contracts by $\frac{C}{k}$

$\Rightarrow D_{k \rightarrow e}$ contracts by $1 - \frac{(k-e)}{(k)}$

- Spectral Independence: This for $\chi^2$ and $C=O(1)$
- Entropic Independence: This for $D_{kL}$ and $C=O(1)$

$\Rightarrow$ Poly-time sampling via $D_{k \rightarrow k-0(1)}$

How to prove $D_{0 \rightarrow 1}$ contracts?

This is the difficult part.

We will see many techniques for it:

- In the $\chi^2$ case we want to bound $\lambda_2(U_{i \rightarrow \kappa D_{k \rightarrow 1}})$:

  $$i \rightarrow \begin{pmatrix} 1 & \text{P}[j \text{ is l i e s}] \\ k \text{ convenient} \end{pmatrix}$$

  the matrix we need to analyze
Let \( \psi_{ij} := \Pi[j:i] \) and \( i \) on diagonal.

**Claim:** \( \chi_2(\psi) \leq c \) if and only if

\[
\lg g_\mu \left( z_1^{1/2}, \ldots, z_n^{1/2} \right) \text{ is concave at } z=1.
\]

\[
\left( \nabla^2 \lg g_\mu \left( z_1^{1/2}, \ldots, z_n^{1/2} \right) \right)_{z=1} > 0
\]

**Proof:** Homework calculations. ☀️ □

We will prove \( g_\mu \) is half-plane-stable

\[ \Rightarrow \lg g_\mu \text{ is concave over } \mathbb{R}_0^n. \]

[Gröndahl]

**Toy Case:** Suppose \( p \) is univariate poly with roots \( \in \mathbb{R}_{\leq 0} \). Then \( \lg p \) is concave over \( \mathbb{R}_{>0} \).

\[
\lg p = c + \lg (z+x_1) + \cdots + \lg (z+x_n)
\]

\[
\text{each one concave}
\]

**Note:** This is not \( \lg \)-concavity of \( \text{coeffs of } p \), but it is related!

\text{Lg-concave polynomial: } \lg g_\mu \text{ concave over } \mathbb{R}_0^n.

Another name: Lorentzian