Review

- Barvinok's Method

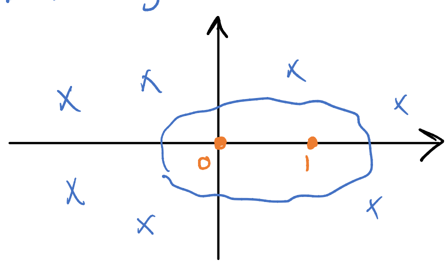
$$p(z) = c_0 + c_1 z + \cdots + c_n z^n$$

* Low coeffs / derivatives at 0 known
* Value at 1 wanted (approximately)

Thm: If simply connected region $U \ni 0, 1$ (say fixed or changing controllably with n) exists where $p(z) \neq 0$ on $z \in U$,

$O(\lg(\frac{n}{\varepsilon}))$ coeffs enough for $(1+\varepsilon)$-approx.

depends on region



- For U = Disk centered at 0 do

Truncated Taylor Series for $\lg p(z)$

- For other U, find polynomial map Φ

$$\Phi(0) = 0 \quad \Phi(1) = 1 \quad \Phi(\text{disk of radius } 1+\delta) \subseteq U$$

and do the same to $p \circ \Phi$.

Φ exists because:

- Riemann mapping gives non-poly Φ from disk to slightly shrunk U.
- Replace by Taylor series to make poly.
- Combine with linear map to get $\Phi(0)=0$ & $\Phi(1)=1$.

- Matching polynomial $p_G(z) = \sum m_k z^k$ (m_k: k-matchings)

Thm: Roots of p_G are real and $\leq -\Omega(\frac{1}{\Delta})$ (Δ: max deg)

Today:
- Trick to get FPTAS
- Real-rootedness & lg-concavity
- Intro to HDX view

For any fixed $z \in \mathbb{R}_{\geq 0}$, $p(z)$ on Δ-bounded-degree graphs has QFPTAS.

How do we remove the Q?

- Smarter way to compute

  $m_k = \#k\text{-matchings}.$

  * Brute-force: $n^{O(k)}$
  * [Patel-Regts]: $\mathrm{poly}(n) \cdot \Delta^{O(k)}$.
- We only go up to $k \simeq \lg n$:)
- Works with

  $\mathrm{ind}(H, G)$: # induced copies of H in G.

  H: fixed graph

- Example: ind( , G) = 2 # edges in G.

We will work with functions

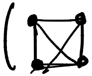
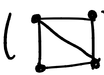
$$f : \text{Graphs} \to \mathbb{C}$$

that can be written as finite linear combination:

$$f(G) := c_1 \,\mathrm{ind}(H_1, G) + \cdots + c_k \,\mathrm{ind}(H_k, G)$$

Example:

- $f(G) = \#\text{vertices} = \mathrm{ind}(\bullet, G)$
- $f(G) = \#\text{edges} = $ ind( , G)/2
- $f(G) = \#2\text{-matchings} =$

  [ind() · 3 + ind() · 2 + ···] / 24

  ⋮

The space of these functions is rich.

- $f, g \longmapsto f+g$
- $f, g \longmapsto f \cdot g$

This is because

$$\mathrm{ind}(H_1, G) \cdot \mathrm{ind}(H_2, G) =$$

$$c_1 \mathrm{ind}(H_1 \geqq H_2, G) + c_2 \mathrm{ind}(H_1 \not\leqq H_2, G) \pm \cdots$$

Call a function additive if

disjoint union.

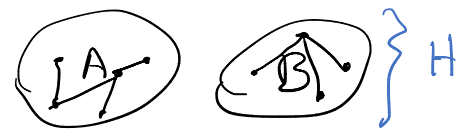
$$f(G_1 + G_2) = f(G_1) + f(G_2).$$

Theorem: Additive functions only need connected graphs H_i in

$$f(G) = c_1 \mathrm{ind}(H_1, G) + \cdots + c_k \mathrm{ind}(H_k, G)$$

Proof: Note that if H is connected $\mathrm{ind}(H, \cdot)$ is additive.

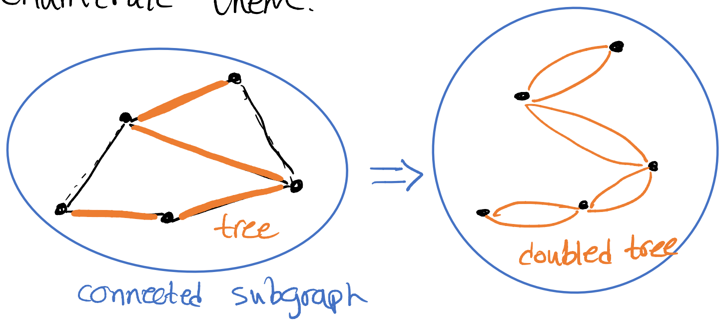
- Suppose $f = \sum c_i \mathrm{ind}(H_i, G)$
- Pick the disconnected H_i with the smallest # of edges.



- Then $f(H_i) - f(A) - f(B) = c_i - 0 - 0$
- This means $c_i = 0$:)

Key Observation: If H is connected, counting $\mathrm{ind}(H, G)$ can be done in $\mathrm{poly}(n)\, \Delta^{O(|H_i|)}$ time.

Proof: There are $n \cdot \Delta^{O(k)}$ many connected induced subgraphs & we can efficiently enumerate them.



The doubled tree is Eulerian: $\exists$ tour of length $2(k-1)$.

$\Rightarrow$ conn. ind subgraphs $\subseteq$ subgraphs induced by vertices of length $2(k-1)$ walks.

$\# \leq n \cdot \Delta^{2(k-1)} = n \cdot \Delta^{O(k)}$

Thm: Additive $f = \sum c_i \,\mathrm{ind}(H_i, G)$ with $k = \max_i |H_i|$. Naive ALG computes f on G of size $\leq k$.

$\Rightarrow \exists$ ALG' for $f(G)$. runtime $= n \cdot \Delta^{O(k)}$.

Proof: We can ignore all H_i not subgraph of G.

Enumerate all induced subgraphs $H_1, \dots, H_m$ $\leq n \cdot \Delta^{O(k)}$

Sort from fewest to most # edges.

$c_1 = f(H_1)$

$c_2 = f(H_2) - c_1 \,\mathrm{ind}(H_1, H_2)$

$c_3 = f(H_3) - c_1 \,\mathrm{ind}(H_1, H_3) - c_2 \,\mathrm{ind}(H_2, H_3)$

$\vdots$

Once we know all c_i, we simply compute $\mathrm{ind}(H_i, G)$ and take

$$f(G) = c_1 \,\mathrm{ind}(H_1, G) + \dots + c_m \,\mathrm{ind}(H_m, G).$$

How to use for # k-matchings?

not additive

Key Idea: The matching poly is multiplicative.

$$m_k(G_1+G_2) =$$

$$m_0(G_1)m_k(G_2) + m_1(G_1)m_{k-1}(G_2) + \cdots + m_k(G_1)m_0(G_2)$$

$$\Rightarrow P_{G_1+G_2}(z) = P_{G_1}(z) \cdot P_{G_2}(z)$$

So root moments are additive:

$$G \longmapsto \lambda_1^\alpha + \cdots + \lambda_n^\alpha \quad \text{with } \{\lambda_i\} = \text{roots}$$

for any fixed α.

Newton's Identities: Moments $\alpha = 0, -1, \ldots, -k$ are polynomials of $p^{(0)}(0), \ldots, p^{(k)}(0)$ and vice versa.

$$\sum \lambda_i^0 = n$$

$$\sum \lambda_i^{-1} = \frac{-p^{(1)}(0)}{p^{(0)}(0)} = -p^{(1)}(0)$$

$$\sum \lambda_i^{-2} = \left(\sum \lambda_i^{-1}\right)^2 - 2\sum_{i<j} \lambda_i \lambda_j = \left(p^{(1)}(0)\right)^2 - 2p^{(2)}(0)$$

$$\vdots$$

ALG:

- Compute $\alpha = -k, \ldots, 0$ moments of G.
  - * Each only needs $\leq$ size $O(k)$ H_i
  - * Each is additive
- Use reverse Newton identities to get first k coeffs.

For $k \approx \lg(\frac{n}{\varepsilon})$ takes time $\text{poly}(n) \cdot \Delta^{\lg(\frac{n}{\varepsilon})}$, so FPTAS. :)

Real-Rootedness & Log-Concavity

We saw p_G has real roots.

matching poly

Thm: $m_k^2 \geq m_{k-1} m_{k+1}$ we needed this before, remember?

Proof:

$$p \text{ real-rooted} \Rightarrow \begin{cases} p' \text{ real-rooted} & \text{mean-value} \\ z^{d} p(\tfrac{1}{z}) \text{ real-rooted} & \end{cases}$$

degree

roots inverted

$$p \Rightarrow q := p^{(k-1)} \Rightarrow r = z^d q(\tfrac{1}{z}) \Rightarrow$$

$$s = r^{(d-2)} = a + bz + cz^2$$

$a = m_{k+1} \cdot \frac{(k+1)!}{2} \cdot (d-2)!$

$b = m_k \cdot k! \cdot (d-1)!$

$c = m_{k-1} (k-1)! \cdot d!/2$

$b^2 \geq 4ac \Rightarrow$

$$m_k^2 \geq m_{k-1} \cdot m_{k+1} \cdot \frac{d}{d-1} \cdot \frac{k+1}{k} \quad \geq 1$$

$\square$

Remark: One can see this is proving if $c_0 + c_1 z + \cdots + c_n z^n$ is real-rooted, then

$$\frac{c_0}{\binom{n}{0}}, \frac{c_1}{\binom{n}{1}}, \ldots, \frac{c_n}{\binom{n}{n}} \text{ is lg-concave.}$$

Remark: The reverse is false.

We now study another real-rooted poly.

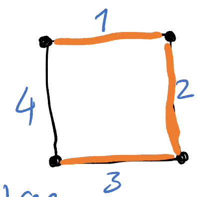
this time multivariate

Real-rootedness is the "dream" in counting & sampling.

Spanning Tree Polynomial

$p_G(z_1, z_2, z_3) =$

$z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2$



each term a spanning tree

- p_G is homogeneous of deg $= n-1$
- It has m variables

Thm: $p_G = \det(z_1L_1 + \dots + z_mL_m)$ where L_i is the Laplacian of edge i with row/col n removed.

Proof: This is the matrix-tree thm. Think of z_i as an integer counting parallel edges. □

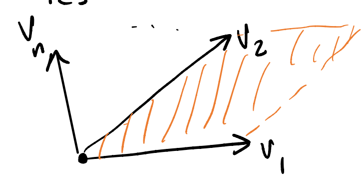
More generally look at

$$p := \det(z_1A_1 + \dots + z_nA_n)$$

where $A_i \succeq 0$. These define for rank-1 A_i determinantal point processes (DPPs).

*technically a special case of DPPs.

$$p = \sum_S \mu(S) \prod_{i \in S} z_i$$



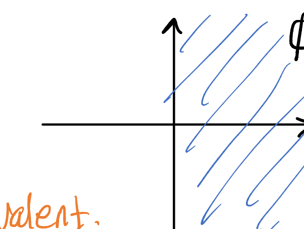
$A_i = v_iv_i^T$

$v_i \in \mathbb{R}^k$

$$\mu(S) \propto \mathrm{vol}(\{v_i\}_{i \in S})^2 \quad \text{for } |S| = k$$

Thm: Whenever $\mathrm{Re}(z_i) > 0$ we have

$$\det\left(\sum z_iA_i\right) \neq 0$$



Note: Any half-plane equivalent.

We will show a standard Markov chain called down-up walk mixes rapidly for DPPs & half-plane-stable dists.

High Dimensional Expanders (HDX)

We will study these in detail, but for now a distribution μ on $\binom{[n]}{k}$.

think weighted hypergraph

- Spanning trees: $n = \#$ edges in G
  $k = \#$verts -1 in G

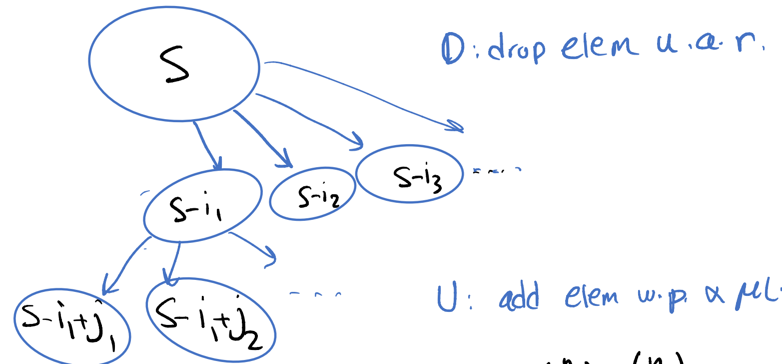
- Noise operator $D \in \mathbb{R}^{\binom{n}{k} \times \binom{n}{k-1}}$

down

$$D(S,T) = \begin{cases} 0 & \text{if } T \not\subset S \\ \frac{1}{k} & \text{if } T \subset S \end{cases}$$

- We call the time-reversal D° the up operator U.

- Down-up walk: $DD^\circ = DU$.



- More generally $D_{k\to\ell} \in \mathbb{R}^{\binom{n}{k} \times \binom{n}{\ell}}$ maps $S \in \binom{[n]}{k}$ to unif. random $T \in \binom{S}{\ell}$.

- We call $D_{k\to\ell}^\circ$ alternatively $U_{\ell\to k}$ ← μ-specific

Inf. Def. μ is HDX when $D_{k\to 1}$ contracts χ^2-divergence.

Plan: μ = DPPs / spanning trees

1. $P_\mu(z_1, \dots, z_n) \neq 0$ for $\mathrm{Re}(z_i) > 0$.

$\Downarrow$

2. Show μ is an HDX.

$\Downarrow$

3. Conclude fast mixing of DU.

Thm: For any $p := \det(\sum z_i A_i)$ with $A_i \succeq 0$ we have $p \neq 0$ for $\mathrm{Re}(z_i) > 0$.

Proof: We take $A_j \succ 0$ and take limits.

Assume $z_j = a_j + i \cdot b_j$. Then

$$\sum z_i A_i = \underbrace{\left(\sum a_j A_j\right)}_{A} + i \underbrace{\left(\sum b_j A_j\right)}_{B}$$

A is symmetric P.S.D.

$$\det(A + iB) = \det(A) \cdot \det\left(I + i A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)$$

$$= i^k \det(A) \det\left(A^{-1/2} B A^{-1/2} - iI\right)$$

For this to be zero i has to be eigenvalue of $\underbrace{A^{-1/2} B A^{-1/2}}_{\text{Symmetric}}$. But Symmetric matrices have real eigs.

$\square$

For Fun Corollary:

Let a_k = # s.trees with $\deg(v) = k$.

Then a_k is log-concave:

$$a_k^2 \geq a_{k-1} \cdot a_{k+1}.$$

Proof: Plug $z_e \leftarrow z$ for $e \sim v$ and $z_e \leftarrow 1$ for $e \not\sim v$. We get real-rooted poly.

Any $z \notin \mathbb{R}_{\leq 0}$ lies in a half-plane with 1.