Review

- Barvinok's Method
  \[ p(z) = c_0 + c_1 z + \cdots + c_n z^n \]
  * Low coeffs / derivatives at 0 known
  * Value at 1 wanted (approximately)
  
  \[ O(\log(\frac{1}{z})) \] coeffs enough for \((k\epsilon)\)-approx.

Thm: If simply connected region \( U \ni 0, 1 \)

say fixed or changing controllably with \( n \)
exists where \( p(z) \neq 0 \) on \( z \in U \),

\[ O(\log(\frac{1}{z})) \] coeffs enough for \((k\epsilon)\)-approx.

depends on region

\[ \Re \]
For any fixed $z \in \mathbb{R}_{>0} \setminus p(\mathbb{R})$, on $\Delta$-bounded-degree graphs has QFPTAS.

How do we remove the $\mathcal{O}$?

- Smarter way to compute
  $$m_k = \#k\text{-matchings}.$$  

- Brute-force: $n \mathcal{O}(z)$
- \textbf{[Patel–Regts]}: $\text{poly}(n) \cdot A$

- We only go up to $k = \lg n \pm$.
- Works with $\text{ind}(H, G)$: \#induced copies of fixed graph $H$ in $G$.

Example: $\text{ind}(\bigstar, G) = \#\text{edges in } G$.

We will work with functions $f: \text{Graphs} \rightarrow \mathbb{C}$ that can be written as finite linear combination:

$$f(G) = c_1 \text{ind}(H_1, G) + \cdots + c_k \text{ind}(H_k, G)$$

Example:

- $f(G) = \#\text{vertices} = \text{ind}(\bigstar, G)$
- $f(G) = \#\text{edges} = \text{ind}(\bigcirc, G)/2$
- $f(G) = \#2\text{-matchings} = \left[\text{ind}(\bigcirc) \cdot 3 + \text{ind}(\Box) \cdot 2 + \cdots \right]/24$
The space of these functions is rich.

- $f, g \mapsto f + g$
- $f, g \mapsto f - g$

This is because

$$\text{ind}(H_1, G) \cdot \text{ind}(H_2, G) =$$

$$c_1\text{ind}(H_1 \supset H_2, G) + c_2\text{ind}(H_1 \subseteq H_2, G) + \cdots$$

Call a function additive if

disjoint union

$$f(G_1 \cup G_2) = f(G_1) + f(G_2).$$

Theorem: Additive functions only need connected graphs $H_i$ in

$$f(G) = c_1\text{ind}(H_1, G) + \cdots + c_k\text{ind}(H_k, G)$$

Proof: Note that if $H$ is connected

$$\text{ind}(H, \cdot)$$

is additive.

- Suppose $f = \sum c_i \text{ind}(H_i, G)$
- Pick the disconnected $H_i$ with the smallest # of edges.
- Then $f(H_i) - f(A) - f(B) = c_i \cdot 0 - 0$
- This means $c_i = 0$ 😊

Key Observation: If $H$ is connected, counting $\text{ind}(H, G)$ can be done in $\text{poly}(n) \Delta O(1 |H|)$ time.
Proof: There are $n \cdot \Delta^0(n)$ many connected induced subgraphs $S$ we can efficiently enumerate them.

The doubled tree is Eulerian: $f$ a tour of length $2(k-1)$.

$\Rightarrow$ connected subgraphs $\subseteq$ subgraphs induced by vertices of length $2(k-1)$ walks.

$\# \leq n \cdot \Delta^0(n) = n \cdot \Delta^0(n)$

$$\text{Proof: (continued)}$$

$\textbf{Thm:}$ Additive $f = \sum c_i \text{ind}(H_i;G)$ with $\kappa = \max |H_i|$. Naive ALG computes $f$ on $G$ of size $\leq \kappa$.

$\Rightarrow$ JALG' for $f(G)$, runtime $= n \cdot \Delta^0(n)$.

$\textbf{Proof:}$ We can ignore all $H_i$ not subgraph of $G$.

Enumerate all induced subgraphs $H_1, \ldots, H_m$.

Sort from fewest to most edges.

$c_1 = f(H_1)$

$c_2 = f(H_2) - c_1 \text{ind}(H_1;H_2)$

$c_3 = f(H_3) - c_1 \text{ind}(H_1;H_3) - c_2 \text{ind}(H_2;H_3)$

$\vdots$

Once we know all $c_i$, we simply compute

$\text{ind}(H_i;G)$ and take

$$f(G) = c_1 \text{ind}(H_1;G) + \ldots + c_m \text{ind}(H_m;G).$$
How to use for \# k-matchings?

**Key Idea:** The matching poly is multiplicative.

\[
m_k(G_1 \oplus G_2) = m_0(G_1)m_k(G_2) + m_1(G_1)m_{k-1}(G_2) + \cdots + m_k(G_1)m_0(G_2)
\]

\[ \Rightarrow \quad P_{G_1 \oplus G_2}(z) = P_{G_1}(z) \cdot P_{G_2}(z) \]

So root moments are additive:

\[ G \mapsto \lambda_1^x + \cdots + \lambda_n^x \] with \( \{\lambda_i\} \) roots for any fixed \( x \).

**Newton's Identities:** Moments \( \alpha = 0, -1, \ldots, -k \) are polynomials of \( p^{(0)}(0), \ldots, p^{(k)}(0) \) and vice versa.

\[
\sum \lambda_i^0 = n
\]
\[
\sum \lambda_i^{-1} = -\frac{p^{(1)}(0)}{p^{(0)}(0)} = -p^{(1)}(0)
\]
\[
\sum \lambda_i^{-2} = (\sum \lambda_i^{-1})^2 - 2\sum_{i<j} \lambda_i \lambda_j = (p^{(1)}(0))^2 - 2p^{(2)}(0)
\]

**ALG:**

- Compute \( \alpha = -k, -1, \ldots, 0 \) moments of \( G \).
  * Each only needs \( \leq \text{size } O(k) \) H1.
  * Each is additive.
- Use reverse Newton identities to get first \( k \) coeffs.

For \( k = \lg \left( \frac{n}{\epsilon} \right) \) takes time \( \Delta \lg \left( \frac{n}{\epsilon} \right) \), so FPTAS. 😊
Real-Rootedness & Log-Concavity

We saw $P_Q$ has real roots.

**Thm:** $m_k^2 \geq m_{k-1}m_{k+1}$ \(\leftarrow\) we needed this before, remember?

**Proof:**

\[
p \text{ real-rooted} \Rightarrow \int p(x) \text{ real-rooted} \quad \text{mean-value}
\]

\[
\int z^d p(\frac{1}{z}) \text{ real-rooted} \quad \text{roots inverted}
\]

\[
p \Rightarrow q := p^{(k-1)}(x) \Rightarrow r := z^d q(\frac{1}{z}) \Rightarrow
\]

\[
s = r^{(d-2)} = a + bZ + cZ^2
\]

\[
a = m_{k+1} \cdot \frac{(k-1)!}{2} \cdot (d-2)!
\]

\[
b = m_k \cdot k! \cdot (d-1)!
\]

\[
c = m_{k-1} \cdot (k-1)! \cdot d! / 2
\]

\[
b^2 \geq 4ac \Rightarrow \quad m_k^2 \geq m_{k-1}m_{k+1} \cdot \frac{d}{d-1} \cdot \frac{k+1}{k}
\]

**Remark:** One can see this is proving if $c_0 + c_1 z + \cdots + c_n z^n$ is real-rooted, then

\[
\frac{c_0}{(n)!} , \frac{c_1}{(n)!} , \cdots , \frac{c_n}{(n)!}
\]

is log-concave.

**Remark:** The reverse is false.

We now study another real-rooted poly.

this time multivariate

Real-rootedness is the "dream" in counting & sampling.
Spanning Tree Polynomial

\[
P_G(z_1 z_2 z_3) = z_1 z_2 z_3 + z_2 z_3 z_4 + z_3 z_4 z_1 + z_4 z_1 z_2
\]

- \( P_G \) is homogeneous of deg = \( n-1 \)
- It has \( m \) variables

**Thm:** \( P_G = \det(z_1 L_1 + \ldots + z_m L_m) \) where \( L_i \) is the Laplacian of edge \( i \) with row/column \( n \) removed.

**Proof:** This is the matrix-tree thm.
Think of \( z_i \) as an integer counting parallel edges.

More generally look at

\[
\rho := \det(z_i A_i \ldots + z_n A_n)
\]

where \( A_i \in \mathbb{R} \). These define for rank-1 \( A_i \) determinantal point processes (DPPs).

Technically a special case of DPPs.

\[
p = \sum_s M(s) \prod_{i \in s} z_i
\]

\( \mu(s) \propto \text{vol}(\bigcup_{i \in s})^2 \) for \( |s| = k \)

**Thm:** Whenever \( \text{Re}(z_i) > 0 \) we have

\[
\det(\sum z_i A_i) \neq 0
\]

Note: Any half-plane equivalent.
We will show a standard Markov chain called down-up walk mixes rapidly for DPPs & half-plane-stable dists.

We will study these in detail, but for now a distribution \( \mu \) on \( (\mathbb{N}^J \setminus k) \).

Think weighted hypergraph.

- Spanning trees: \( n = \# \) edges in \( G \)
  \( k = \# \) verts \(-1\) in \( G \)

- Noise operator \( D_{G}(R^{n \choose k} \times R^{n \choose k-1}) \)
  - Down \( D(S|T) = \begin{cases} 0 & \text{if } T \subseteq S \\ \frac{1}{k} & \text{if } T \not\subseteq S \end{cases} \)

High Dimensional Expanders (HDX)

We will show a standard Markov chain called down-up walk mixes rapidly for DPPs & half-plane-stable dists.

- We call the time-reversal \( D^0 \) the up operator \( U \).
- Down-up walk: \( DD^0 = DU \).

\[ \begin{align*}
S & \\\\downarrow \\\\downarrow \\
S-i_1 & \\\\to \\\\to \\
S-i_2 & \\\\to \\\\to \\
S-i_3 & \\
\end{align*} \]

- More generally \( D_{k \to \ell} \in \mathbb{R}^{n \choose k} \times \mathbb{R}^{n \choose \ell} \) maps \( S \subseteq \binom{[n]}{k} \) to \( \text{unif. random } T \subseteq \binom{[n]}{\ell} \).
- We call \( D_{k \to \ell} \) alternatively \( U_{\ell \to k} \).

Inf. Def. \( \mu \) is HDX when \( D_{k \to 1} \) contracts \( \chi^2 \)-divergence.
Plan: \[ M = \text{OPPs / spanning trees} \]

1. \( p_M(z_1, -i z_n) \neq 0 \text{ for } \text{Re}(z_i) > 0 \)
2. Show \( M \) is an HDX.
3. Conclude fast mixing of DU.

Thm: For any \( p := \det(\sum z_i A_i) \) with \( A_i \notin \mathbb{C} \), we have \( p \neq 0 \text{ for } \text{Re}(z_i) > 0 \).

Proof: We take \( A_j \notin \mathbb{C} \) and take limits.

Assume \( z_j = a_j + i \cdot b_j \). Then

\[
\sum z_i A_i = (\sum a_j A_j) + i (\sum b_j A_j)
\]

A is symmetric PSD.

\[ \det(A + iB) = \det(A) \cdot \det(I + iA^{-\frac{1}{2}}B A^{-\frac{1}{2}}) \]

\[ = i^k \det(A) \det(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} - iI) \]

For this to be zero, \( i \) has to be an eigenvalue of \( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \). But symmetric matrices have real eigs.

For Fun Corollary:

Let \( a_k = \# \text{s.trees with } \deg(v) = k \).

Then \( a_k \) is log-concave:

\[ a_k^2 \geq a_{k-1} \cdot a_{k+1} \]

Proof: Plug \( z = z \) for \( e \in V \) and \( z = 1 \) for \( e \not\in V \). We get real-rooted poly.

Any \( z \notin \mathbb{R} \) lies in a half-plane with \( 1. \)

\[
\begin{pmatrix}
2 & \leftarrow \\
1 & \uparrow \end{pmatrix}
\]