Review

- Comparison Method

  Route ergodic flow of P' through network with capacities = ergodic flow of P.

  Poincaré / LSI for $P' \Rightarrow$ Poincaré / LSI for P.

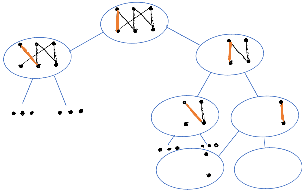
  loss factor: cong × len

  * When len = 1, MLSI transfers too.

  * Often useful for P' = trivial / ideal chain whose ergodic flow is $Q'(x,y) = \mu(x)\mu(y)$.

- Trading time for approx

  Random walk on tree with weights ALG(subprob) mixes in poly(n, α) if ALG is α-approx.



self-reducibility tree

- Canonical Paths

  For an $x \to y$ transition:

  Encoding: $(s,t) \longmapsto (r, \text{side info})$

  another state $\in \Omega$

  from $[M]$.

  pair whose path goes through $x \to y$

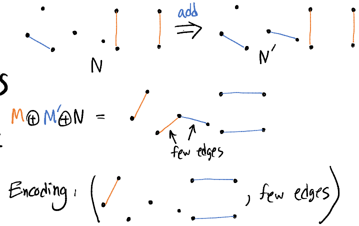
  - Injective
  - $\mu(s)\mu(t) \leq C \cdot \mu(r) Q(x,y)$.

  $\Rightarrow$ Congestion $\leq C \cdot M$.

- Matchings

  We can count / sample in poly(n) time.

  Path: Look at $M \oplus M'$ and traverse paths / cycles one-by-one in some prefixed order.



Plan for Today

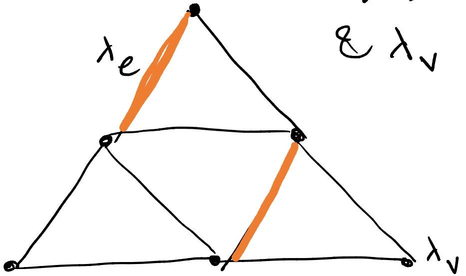
- Matchings ++

- Intro to Correlation Decay

Generalizations

- Monomer-Dimer System:

Weights λ_e on edges & λ_v on verts.



$$\mu(\text{matching } M) = \prod_{e \in M} \lambda_e \prod_{v \notin M} \lambda_v$$

dimer: matched monomer: unmatched

* Remark: λ_v can be "absorbed" in λ_e.

* Thm: $t_{mix} = \text{poly}(n, \max\{\lambda_e\})$ ← assume $\lambda_v = 1$

Proof:

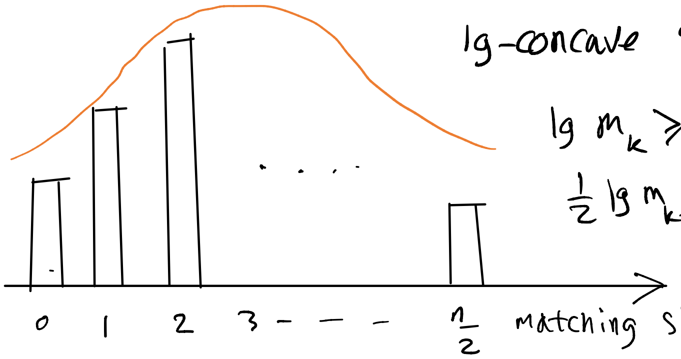
$$\frac{\lambda^M}{Z} \cdot \frac{\lambda^{M'}}{Z} = \underline{\frac{\min\{\lambda^N, \lambda^{N'}\}}{Z}} \cdot \underline{\frac{\lambda^{\text{encoding}(M,M')}}{Z}} \cdot \lambda^{\text{few edges}}$$

$$\mu(M) \cdot \mu(M') \leqslant \text{Poly}(n) \cdot \text{poly}(\lambda) \underbrace{Q(N, N')}_{\gtrsim \frac{1}{\text{Poly}(n)} \min\{\mu(N), \mu(N')\}} \overset{\leq Q(N,N)}{} \mu(\text{encoding})$$

What if we only want perfect matchings?

Idea 1: Define Metropolis Chain on perfect matchings + $\underbrace{\text{near-perfect}}_{\text{size } \frac{n}{2}-1}$ matchings

Idea 2: Look at monomer-dimer with $\lambda_e = \lambda$ a large number.



lg-concave seq.

$$\lg m_k \geqslant \frac{1}{2} \lg m_{k-1} + \frac{1}{2} \lg m_{k+1}$$

matchings of size k ← $m_k^2 \geqslant m_{k-1} \cdot m_{k+1}$

We will prove in future!

- With weight λ:

$$m_k \longmapsto \lambda^k \cdot m_k \qquad \frac{m_{n/2}}{m_{\frac{n}{2}-1}} \leq \frac{m_{\frac{n}{2}-1}}{m_{\frac{n}{2}-2}} \leq \cdots$$

- If $\lambda > \dfrac{m_{n/2-1} \leftarrow \text{near-perfect}}{m_{n/2} \leftarrow \text{perfect}}$

then

$$\lambda^{\frac{n}{2}} m_{\frac{n}{2}} \underset{\text{by choice of } \lambda}{\geq} \lambda^{\frac{n}{2}-1} m_{\frac{n}{2}-1} \overset{\text{by lg-concavity}}{\geq} \cdots \geq \lambda^0 m_0$$

- If we sample from monomer-dimer

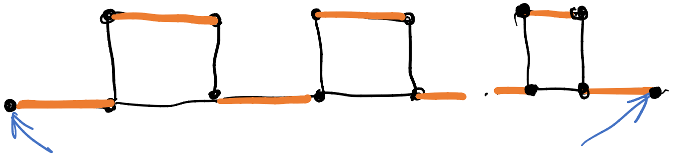
$$\mathbb{P}[\text{perfect matching}] \geq \Omega\left(\frac{1}{n}\right)$$

ALG: Sample & accept if PM

This is poly(n) if $\lambda = \dfrac{m_{n/2-1}}{m_{n/2}} = \text{poly}(n)$.

Remark: Same condition needed for Idea 1.

Bad Example:



- One perfect matching
- $2^{\Omega(n)}$ near-perfect matchings

Thm. In bipartite graphs we can sample perfect matchings. [Jerrum-Sinclair-Vigoda]

Open: For non-bipartite graphs ☹

Idea: Renormalize weights. Let Ω_S be matchings whose monomers are S.

$\lambda : E \to \mathbb{R}_{\geq 0}$

$$w(S) \propto \frac{1}{\sum_{M\in\Omega_S} \lambda^M} .$$

$$\lambda^M = \prod_{e\in M} \lambda_e$$

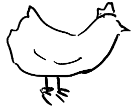
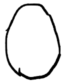
Define $w(M) = w(S)\cdot\lambda^M$ for $M\in\Omega_S$.

- By this choice we have $w(\Omega_S) = w(\Omega_{S'})$.

Thm: Suppose we run Metropolis on $\underbrace{\text{perfect + near-perfect}}_{|S|\in\{0,2\}}$ matchings in a bipartite graph. Then

$$1-\lambda_2(P) \geq \frac{1}{\text{poly}(n)}$$

← poly-time mixing if we start from arg max $w(\cdot)$.

The  +  problem

- How do we come up with $w(S)$?
  ↓ this is counting!

- **Idea:** The thm still holds if $w(S)$ are approx $\underset{\sim}{\propto} 1/\sum_{M\in\Omega_S}\lambda^M$
  say up to factor 10.

- Start with a symmetric easy graph, e.g. $K_{\frac{n}{2},\frac{n}{2}}$ with $\lambda_e = 1$, & slowly change λ_e ← by $1+\frac{1}{n}$ factor

- Use Markov chain to sample & set

$$w'(S) \underset{\sim}{\propto} w(S) / \underbrace{\mathbb{P}[\text{sample} \in \Omega_S]}_{\text{estimate via Monte Carlo}} = w(S) \sum_{M\in\Omega_S} \lambda^M$$

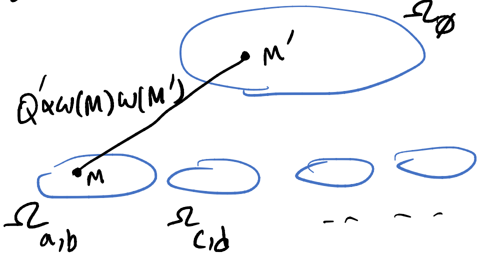
- $\lambda_e = 0 \underset{\text{approx the same}}{\leftrightarrow} \lambda_e = \frac{1}{\exp(n^2)}$

Sketch of Mixing Proof:

- For simplicity we use exact $w(S)$.

everything holds for approx w.

- Enough to route flow from near-perfect $M \in \Omega_{a,b} \to M \in \Omega_{\emptyset}$ perfect with low congestion.



P': star-trivial chain

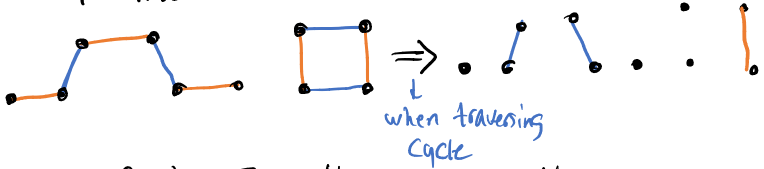
* Equivalently: To route from $\Omega_{a,b}$ to $\Omega_{c,d}$ we first pick a random intermediate $M'' \in \Omega_{\emptyset}$.

- Routing: $M \oplus M'$ is one alternating path and some cycles. We always traverse alternating path first & then the cycles.

Encoding: Same as before (for $N \to N'$ transition) ($M \oplus M' \oplus N$ - few edges, side info)

Issue: Note that encoding might not be perfect / near-perfect. It might have 4 monomers.

$w(\text{encoding}) = \lambda^{M} \cdot \underline{w(4}$



This is fine! Even though encoding $\in \Omega_{a,b,c,d}$

$$w(M)\,w(M') \leq \underbrace{\text{poly}(n) \cdot \min\{w(N)w(N')\}}_{\simeq Q(N,N')} \cdot w(\text{encoding})$$

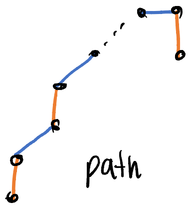
Because the map is still injective

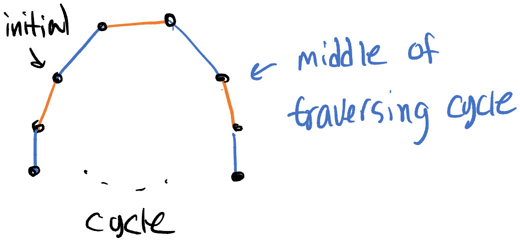
$$\text{congestion} \leq \text{poly}(n) \cdot \frac{\sum_{\text{encodings}} w(N)}{\sum_{M \text{ perfect/near-perfect}} w(M)} = \text{poly}(n).$$

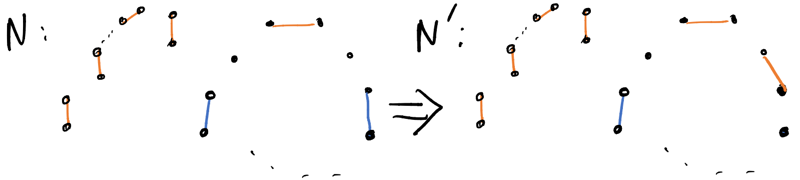
Main Inequality:

$$w(M)\, w(M') \leq \text{poly}(n) \cdot \min\{w(N), w(N')\}\, w(\text{encoding})$$

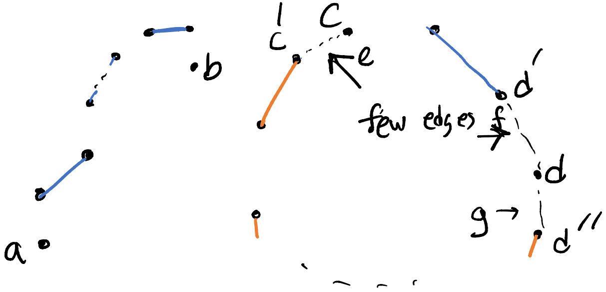
It's a bit of case analysis but let me show you the hardest case!







encoding:



Note that

$$\lambda^{M} \cdot \lambda^{M'} = \lambda^{N} \cdot \lambda^{\text{encoding}} \cdot \lambda_e \cdot \lambda_f$$

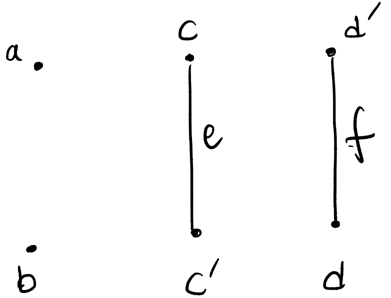
$$\lambda^{M} \cdot \lambda^{M'} = \lambda^{N'} \cdot \lambda^{\text{encoding}} \cdot \lambda_e \cdot \lambda_g$$

To prove inequality we need to show

$$w(\emptyset)\, w(a,b) \leq w(a,b,c,d) \cdot \frac{1}{\lambda_e} \cdot \frac{w(c',d')}{\lambda_f} \cdot \text{poly}(n)$$

$$w(\emptyset)\, w(a,b) \leq w(a,b,c,d) \cdot \frac{1}{\lambda_e} \cdot \frac{w(c',d'')}{\lambda_g} \cdot \text{poly}(n)$$

$$w(\emptyset)\, w(a,b) \leq \text{poly}(n) \times \frac{w(a,b,c,d)\cdot w(c',d')}{\lambda_e \cdot \lambda_f}$$



Proof: Equivalent to show

$$\lambda(\Omega_\emptyset)\,\Omega(\Omega_{a,b}) \geq \frac{1}{\text{Poly}(n)} \cdot \lambda_e \cdot \lambda_f \cdot \underline{\lambda(\Omega_{c',d'})\,\lambda(\Omega_{a,b,c,d})}$$

Idea: Construct maps from $M \in \Omega_{c',d'}$ and $M' \in \Omega_{a,b,c,d}$ to $\Omega_\emptyset \times \Omega_{a,b}$ that are few-to-one & include e & f.

HW problem

Intro to Correlation Decay

Hardcore model:

$$\mu(\text{ind. set } S) \propto \lambda^{|S|}$$

Thm: If $\lambda \leq \frac{1}{\Delta} \Rightarrow$ Dobrushin!

Thm: When $\lambda < \lambda_c(\Delta)\cdot(1-\delta)$ with

$$\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \simeq \frac{e}{\Delta}$$ then

we can still sample/count.

[Weitz: Correlation Decay]

Later in the course we will see Glauber dynamics works!

Thm[Sly]: NP-hard for $\lambda > \lambda_c(\Delta)\cdot(1+\delta)$