Limited Correlations, Fractional Log-Concavity, and Fast Mixing

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based on joint works with

Kuikui Liu

Shayan OveisGharan

Cynthia Vinzant

Efficient sampling/counting/inference when pairwise correlations are . . .

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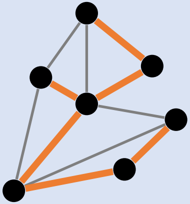
Negative



random spanning tree

$\mathbb{P}[\text{tree } T] \propto 1$

[Feder-Mihail'92, . . .]

$$\mathbb{P}[\text{edge } e \in T \mid \text{edge } f \in T] \leqslant \mathbb{P}[\text{edge } e \in T] \text{ for } e \neq f$$

Efficient sampling/counting/inference when pairwise correlations are ...

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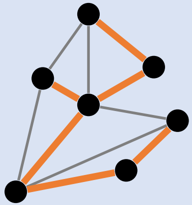

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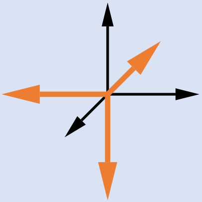

matroid
$\mathbb{P}[\text{basis } B] \propto 1$

[previous talk ...]

spectral bound on matrix of correlations $\equiv$ log-concave polynomial

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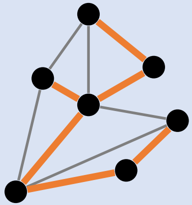

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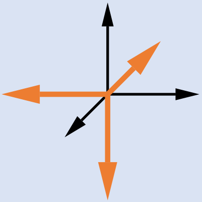

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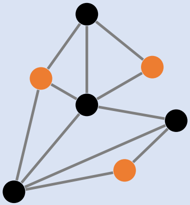


hardcore model

$\mathbb{P}[\text{stable } S] \propto \lambda^{|S|}$

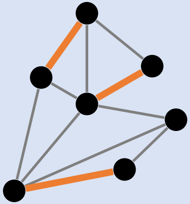
[Weitz'06, ...]

$\mathbb{P}[\text{vertex } u \in S \mid \text{vertex } v \in S] \simeq \mathbb{P}[\text{vertex } u \in S]$ for distant u, v when $\lambda < \lambda_c$

What kind of correlations are useful for sampling/counting/inference?

Example with strange correlations

Monomer-Dimer System



$$\mathbb{P}[\text{matching } M] \propto \prod_{e \in M} \text{weight}(e) \cdot \prod_{v \not\sim M} \text{weight}(v)$$

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- Special case is mystery: random perfect matchings.

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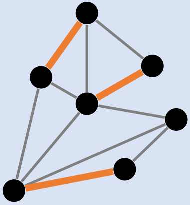

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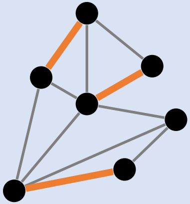

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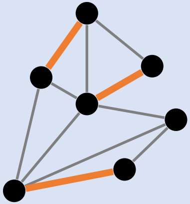

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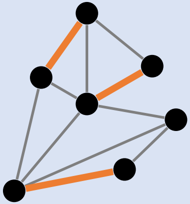

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- Correlations are “limited” . . .

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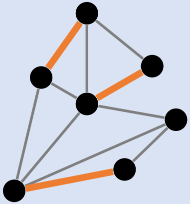

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- Efficient sampling/counting/inference:

spectrally limited correlations

$$\Downarrow$$

fast mixing of simplicial complex walks

Simplicial complex walks

Canonical form

Distribution defined by

$$\mu : \binom{[n]}{k} \to \mathbb{R}_{\geqslant 0}$$

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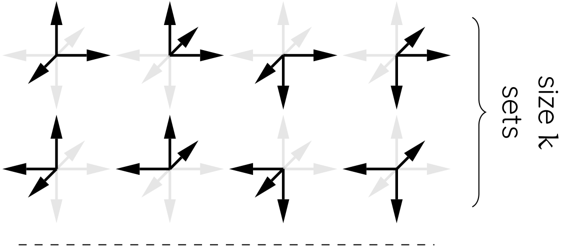

Simplicial complex walks

Canonical form

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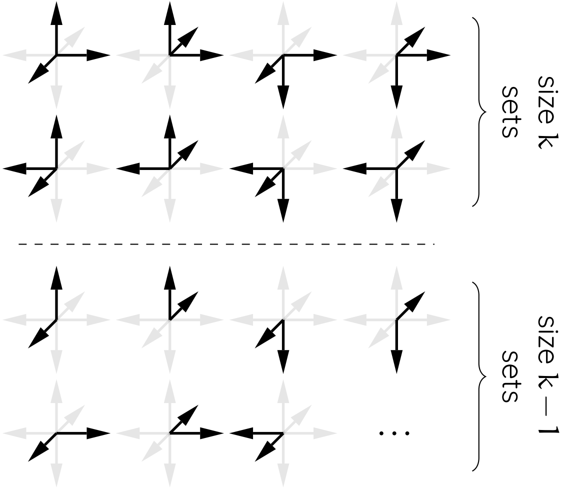
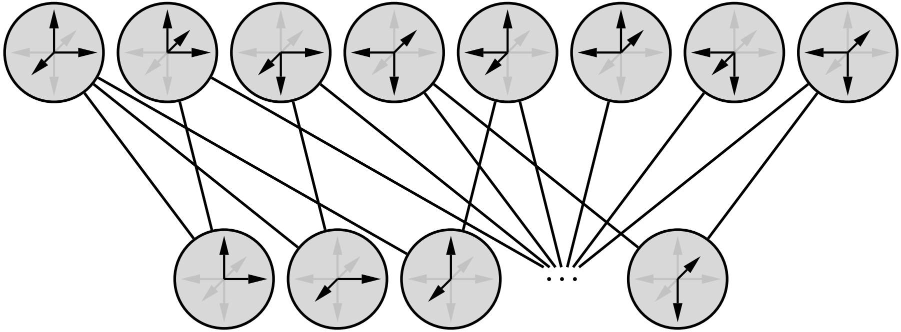
- Matroids conveniently already of this form.
- Natural random walk between $\binom{[n]}{k}$ and $\binom{[n]}{k-1}$.



Random Walk



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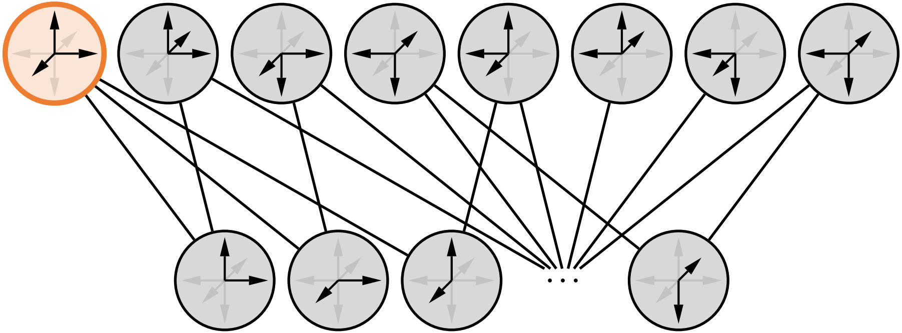

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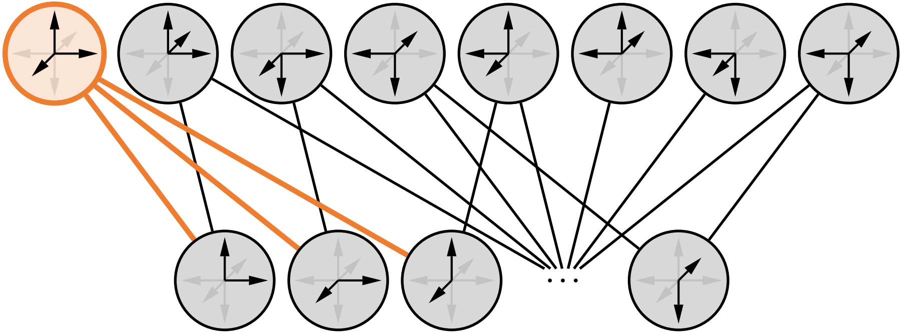
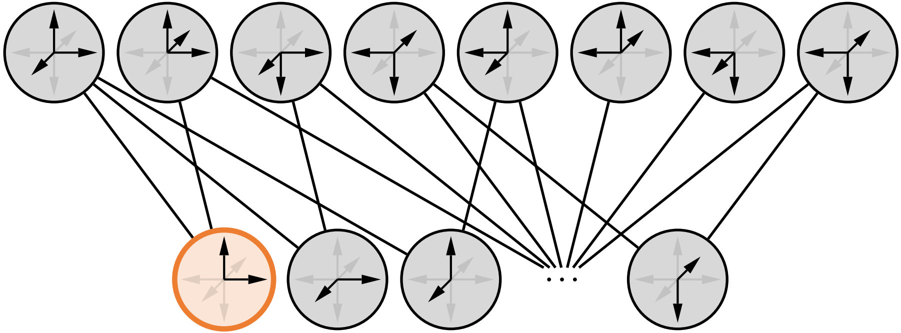


1 Drop an element uniformly at random.

Random Walk



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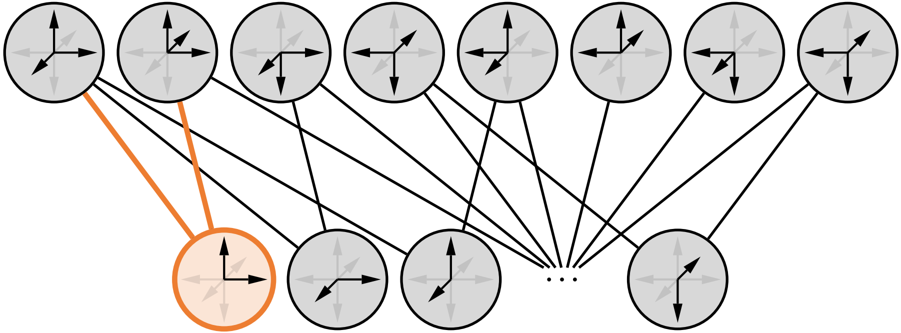

1. Drop an element uniformly at random.
2. Add an element with probability $\propto \mu$(resulting set).

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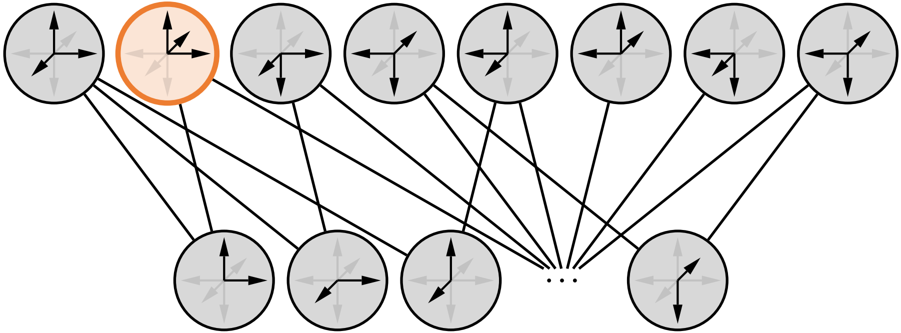

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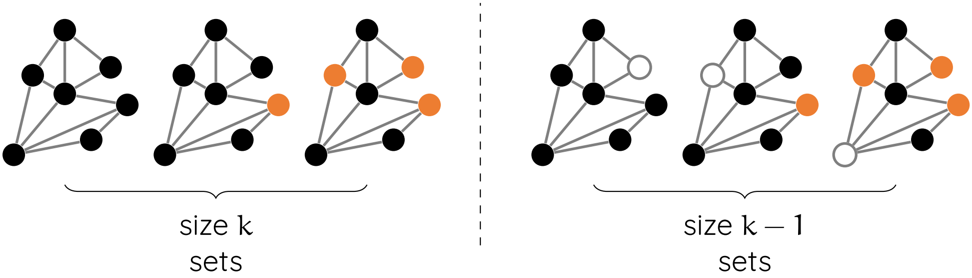

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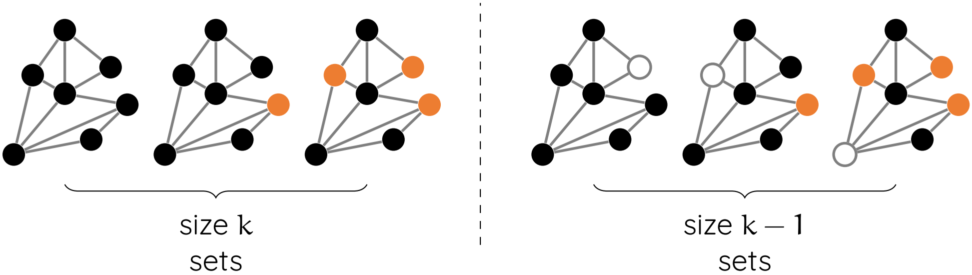

- Random walk is Glauber dynamics:
  1. Unmark vertex uniformly at random.
  2. Make a choice for it with probability $\propto \mu$(resulting configuration).

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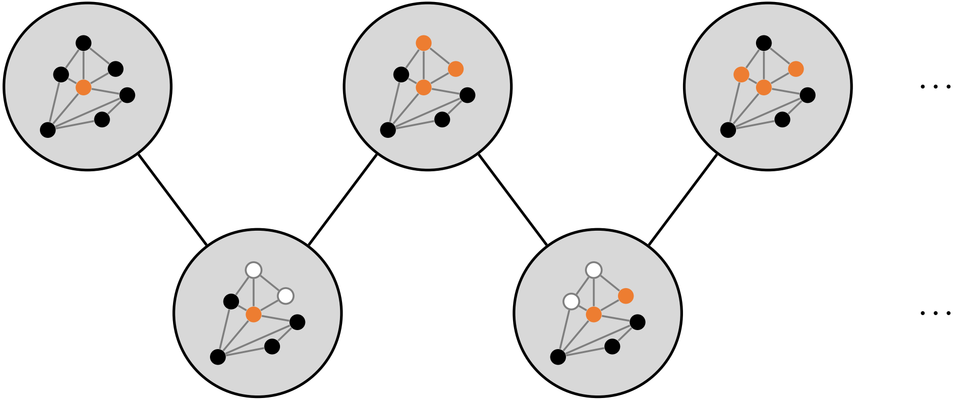
- Random walk between k-sets and $(k-2)$-sets:



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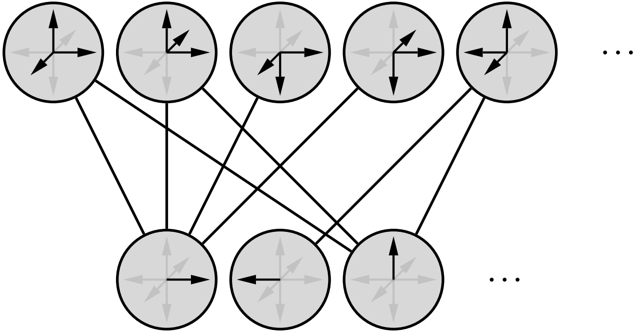

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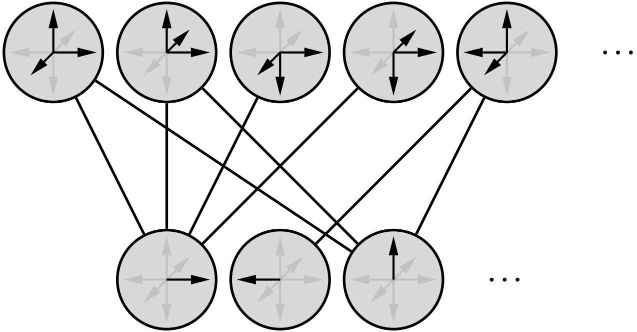

Transition probability matrix from 1-sets to 1-sets:

$$\frac{1}{k}\begin{bmatrix} \vdots & \cdots & \vdots \\ \vdots & \mathbb{P}_{S\sim\mu}[j \in S \mid i \in S] & \vdots \\ \vdots & \cdots & \vdots \end{bmatrix}$$

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- Show that this implies $\mathsf{poly}(k)$ relaxation time for the random walk between k-sets and $(k - O(1))$-sets.

Fractional log-concavity

To our distribution $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geqslant 0}$ we can associate polynomial

$$g(z_1, \ldots, z_n) = \sum_{S \in \binom{[n]}{k}} \mu(S) \prod_{i \in S} z_i.$$

Fractional Log-Concavity

We have $\lambda_{\text{max}}(\text{correlation matrix}) = O(1)$ if and only if

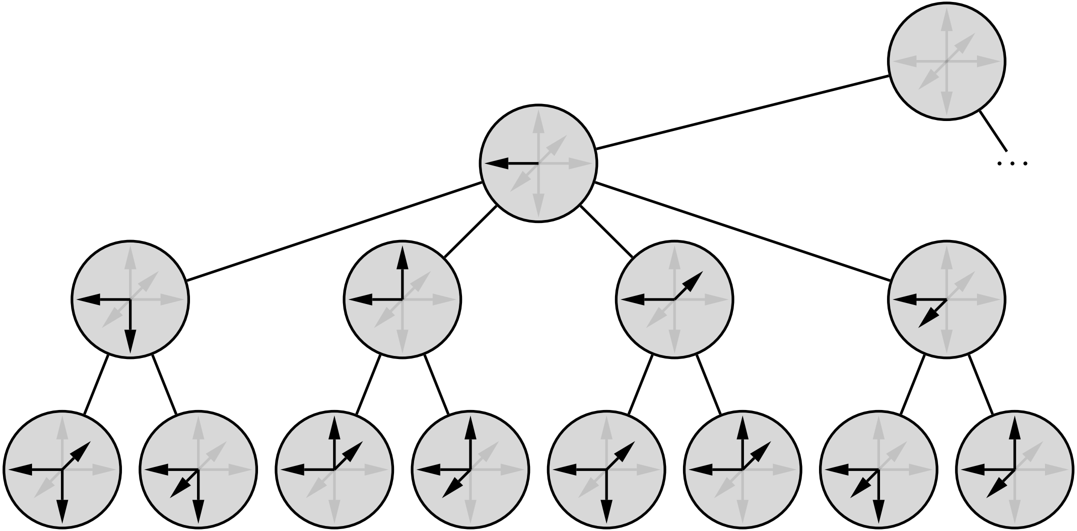
$$g(z_1^\alpha, \ldots, z_n^\alpha)$$

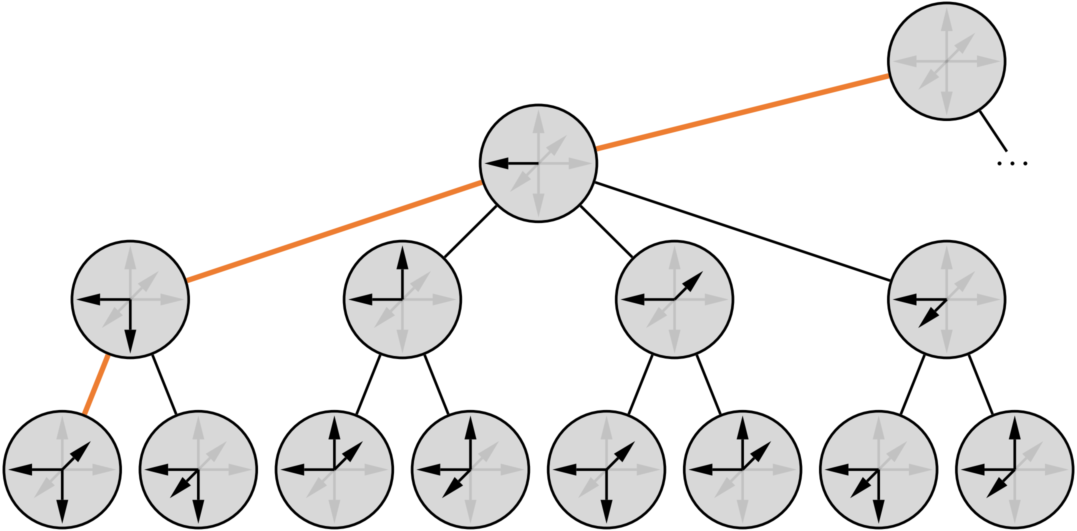
is log-concave around $z_1 = \cdots = z_n = 1$ for some $\alpha = \Omega(1)$.

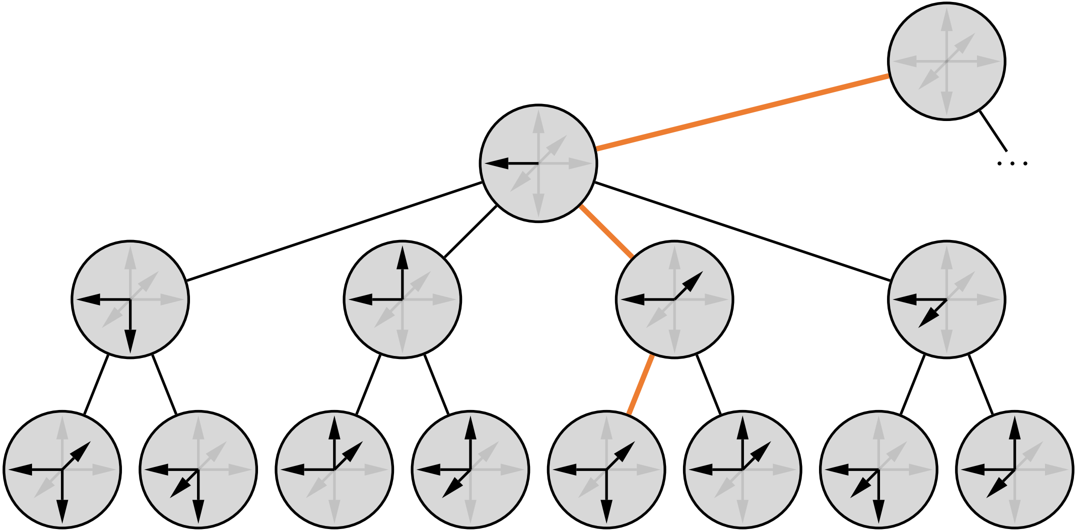
Why does fractional log-concavity imply rapid mixing?

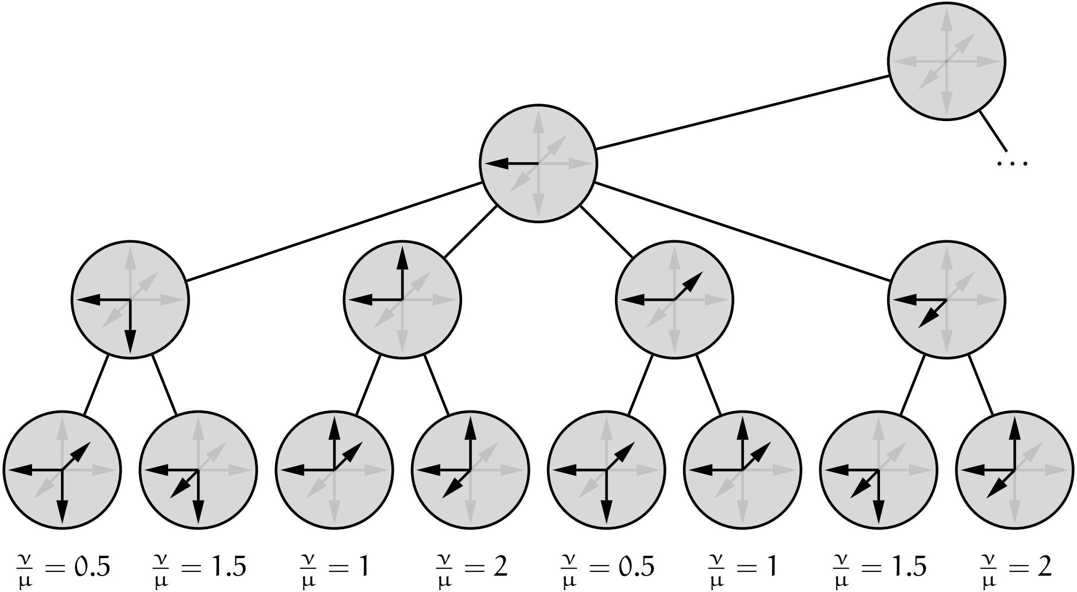
Local-to-global expansion phenomenon

[Kaufman-Oppenheim'17, Cryan-Guo-Mousa'19, Alev-Lau'19, A-Liu-OveisGharan-Vinzant'19].





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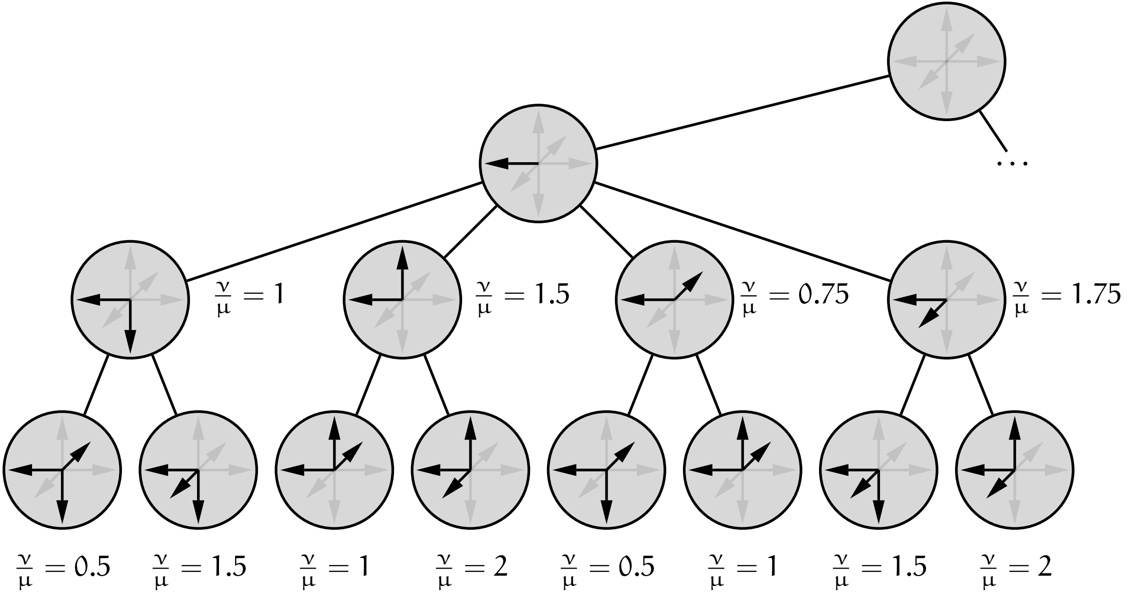

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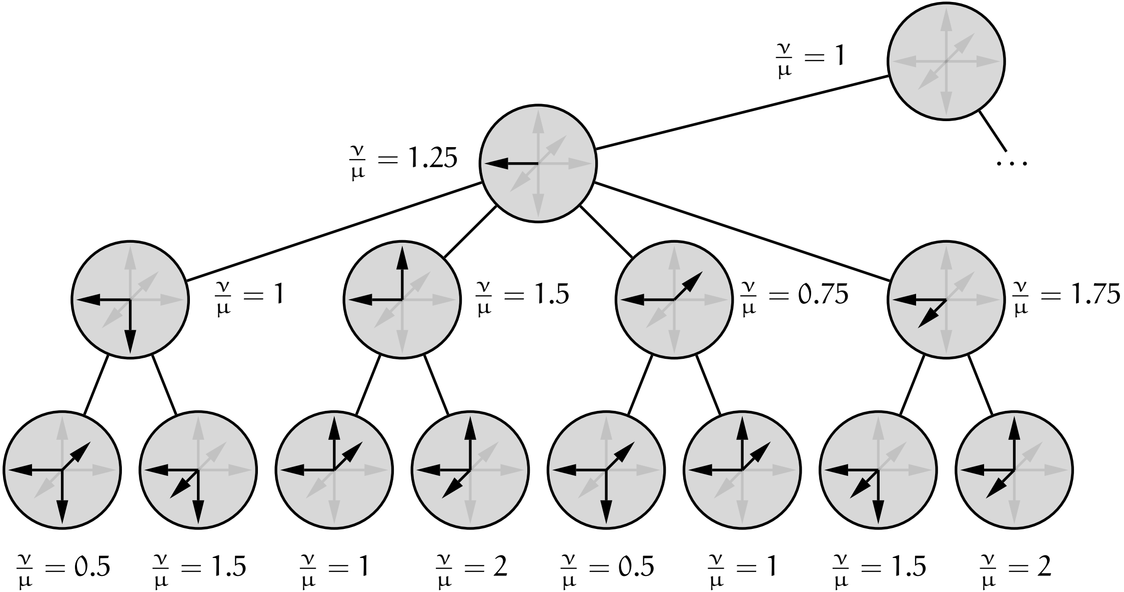
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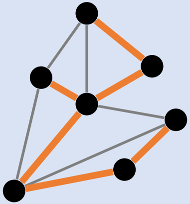

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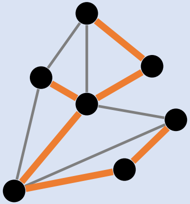

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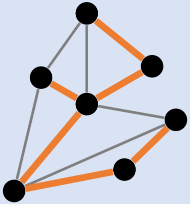

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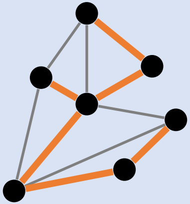

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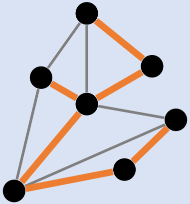

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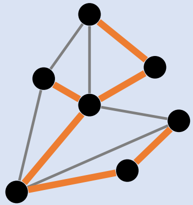

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- This implies $\lambda_{\max} = O(1)$.

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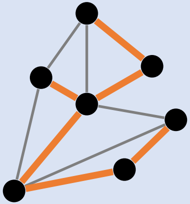

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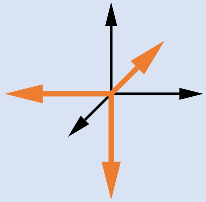
Matroids

Log-Concave Polynomial



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- Previous talk implies

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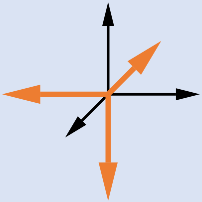

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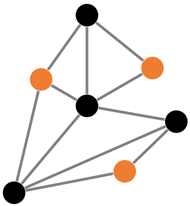
Hardcore Model

Given graph $G = (V, E)$ and $\lambda > 0$, sample a stable set S with

$$\mathbb{P}[S] \propto \lambda^{|S|}.$$



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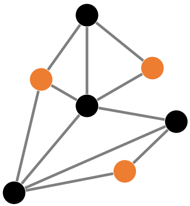

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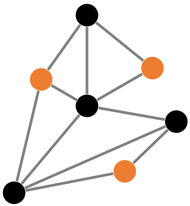

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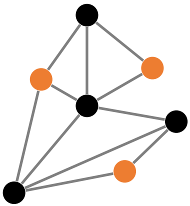

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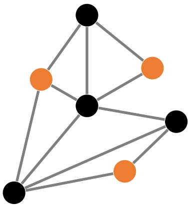

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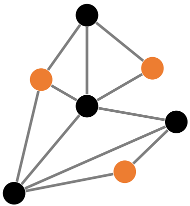

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New Result [A-Liu-OveisGharan'19]

Normal Glauber dynamics mixes in time $n^{f(\epsilon)}$.

- Correlation between u and v decays exponentially

$$|\mathbb{P}[u \mid v] - \mathbb{P}[u]| \leqslant e^{-\Omega(\mathsf{dist}(u,v))}.$$

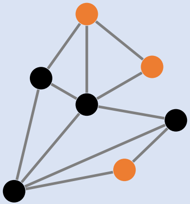
- Use this to bound ℓ_1 norm of columns of correlation matrix

$$\sum_v |\mathbb{P}[u \mid v] - \mathbb{P}[u]|$$

- Unfortunately there are $(\Delta - 1)^d$ nodes at distance d, so not trivial to use decay of correlation.
- Nevertheless, Weitz's self-avoiding walk tree recursion can still be used to show the ℓ_1 norm is $O_\epsilon(1)$.

Monomers

Monomer Distribution



monomer distribution

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#perfect matchings in S^c

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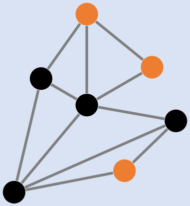

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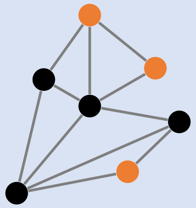

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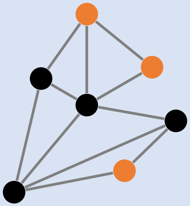

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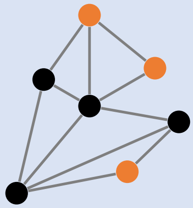

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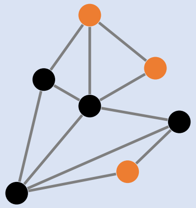

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- This implies efficient sampling for the monomer-dimer systems on planar graphs (hardness shown by [Jerrum'97]).

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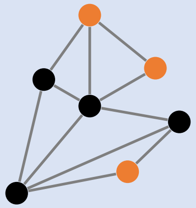

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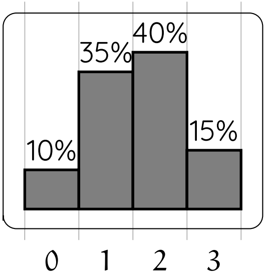

Does this look like #heads dist. in independent (biased) coin flips?

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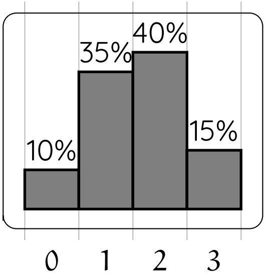

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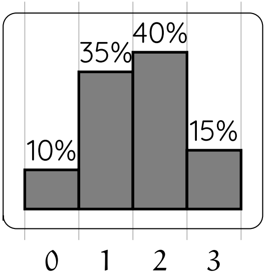

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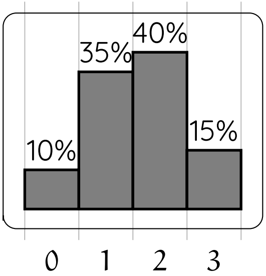
- Roots correspond to biases:

coin bias $p \leftrightarrow 1 - 1/p$ root of g



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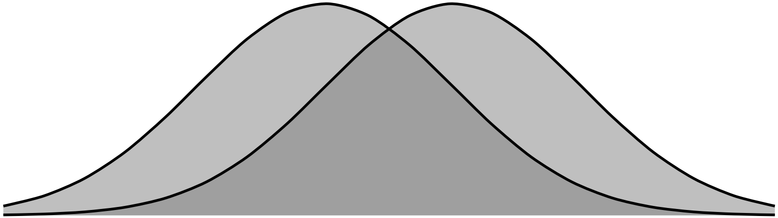

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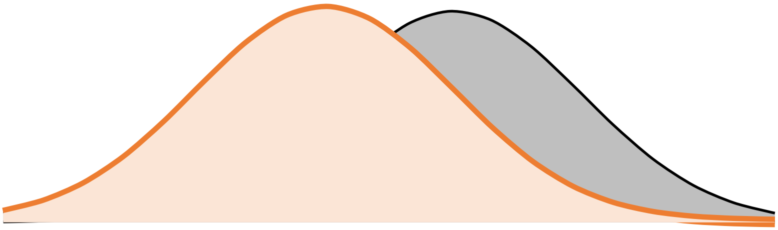

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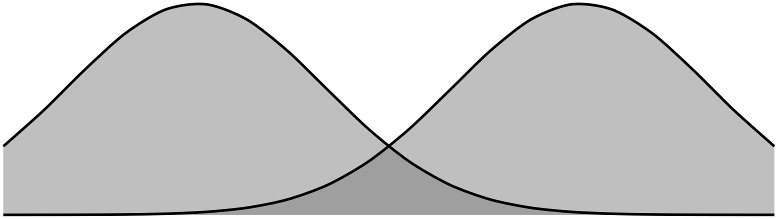

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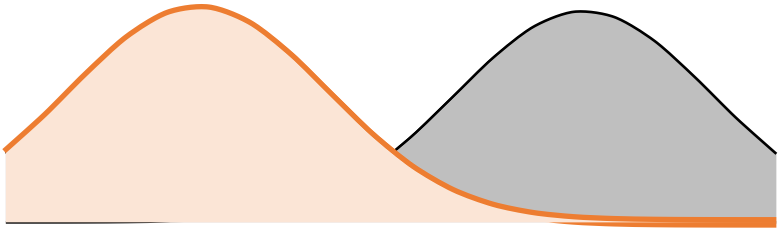

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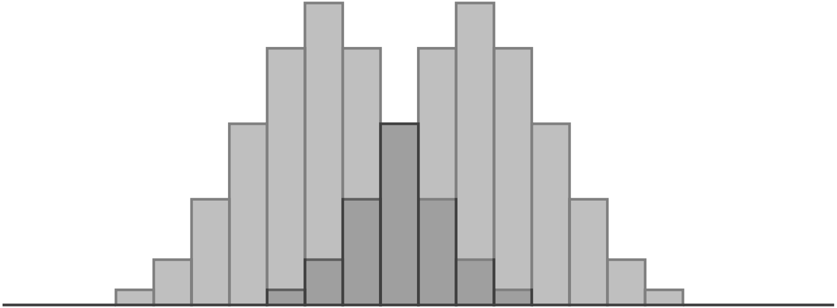

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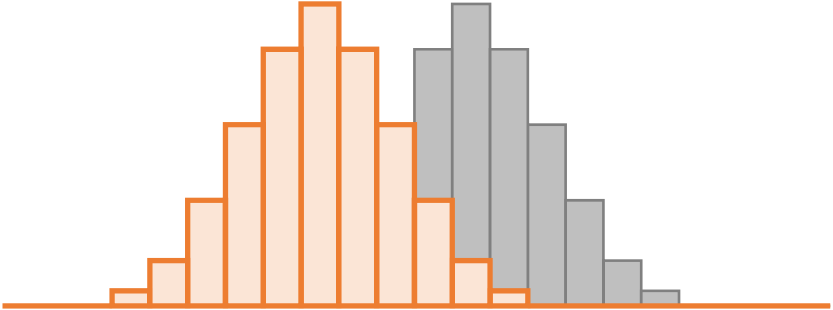

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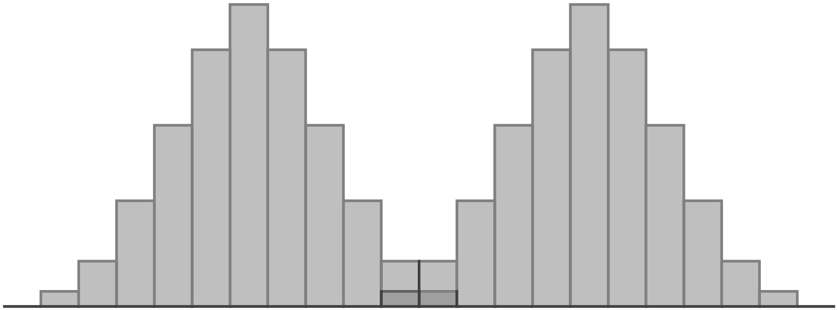

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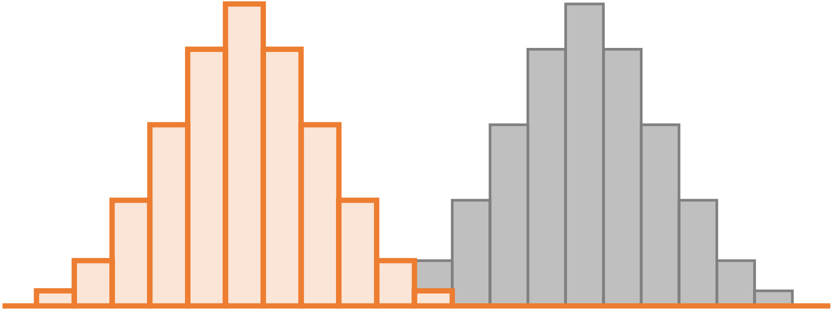

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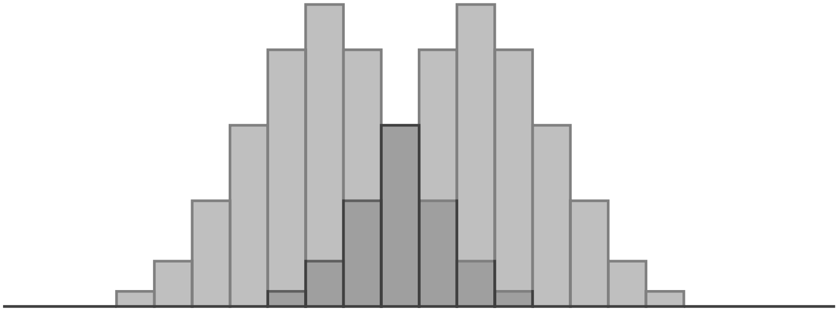
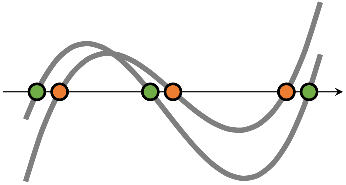


Mixtures of polynomials

[Folklore, used by e.g., MSS'13]

If $\alpha g_1(z) + \beta g_2(z)$ is real-rooted for all $\alpha, \beta > 0$, then roots of g_1, g_2 must have common interlacing.



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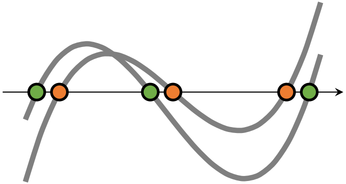

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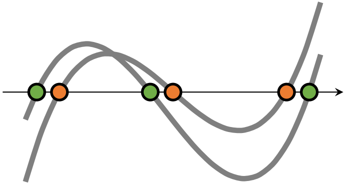

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- Corollary: The means of μ and ν can be off by $\leqslant 1$.

$$|(p_1 + \cdots + p_n) - (q_1 + \cdots + q_n)| \leqslant 1$$

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