CS 260: Geometry of Polynomials in Algorithm Design

Instructor: Nima Anari

Stanford University

Logistics

- Classroom: Wallenberg (Building 160) Room 314
- Time: Tuesdays and Thursdays, 1:30-2:50pm
- Office hours: Tuesdays after class and by appointment

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- Website: `https://nimaanari.com/cs260-winter2020`

  Website is currently empty because of technical difficulties. ☹

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- If you have not officially signed up for the class, but would like to receive announcements, email me at `anari@cs.stanford.edu`.

Evaluation

1. Course project and presentation (teams of up to 2)
2. Brief survey of a paper/papers on a topic
3. Two sets of homework (light)

Letter Grade

One of these combinations:

- 1
- 2 + 3

CR

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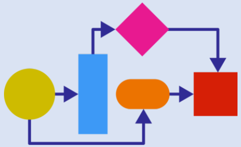
- 1
- 2
- 3

Topics for 1 or 2: suggestions will go on the website; chat with me.

Algorithms



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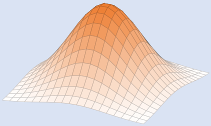
Probability



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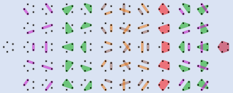
Combinatorics



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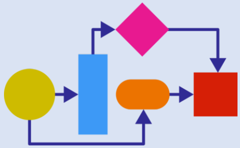

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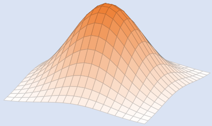

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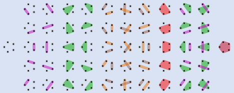
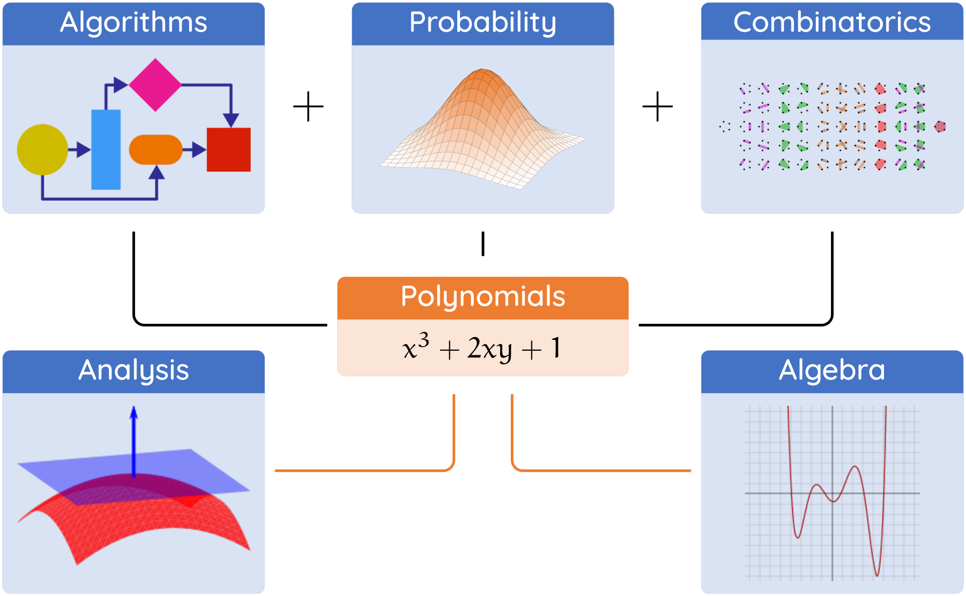


Polynomials

$$x^3 + 2xy + 1$$


Algorithms
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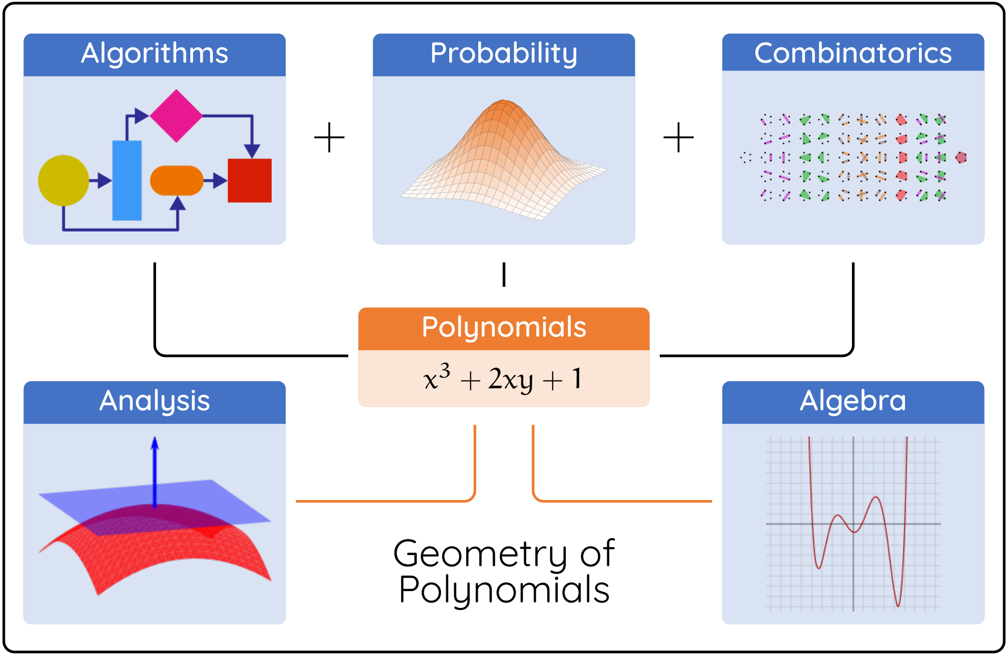
Algorithms
+
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Geometry of Polynomials

Main Paradigm

Different ways of looking at a polynomial:

- $g(z) = a_0 + a_1 z + \cdots + a_d z^d$
- $g(z) = c(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_d)$
- $g : \mathbb{C} \to \mathbb{C}$ or $g : \mathbb{R} \to \mathbb{R}$

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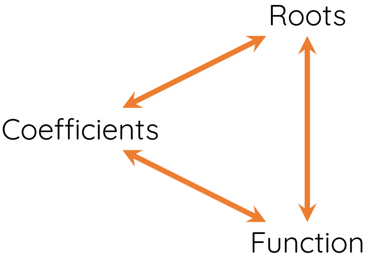
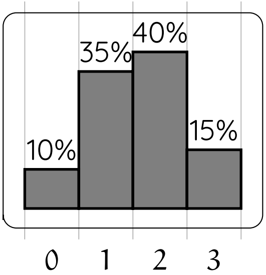


Example: Coin Flips $\leftrightarrow$ Univariate Polynomials



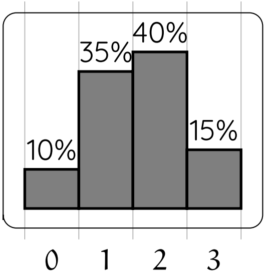
Does this look like #heads dist. in independent coin flips?

Example: Coin Flips $\leftrightarrow$ Univariate Polynomials

Recipe to Verify Coin Flipness

Compute the roots of $g(z)$ and verify that all are real (none are complex).

$$g(z) := 0.10 + 0.35z + 0.40z^2 + 0.15z^3$$



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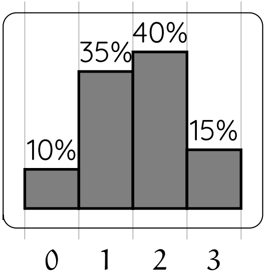
$$g(z) = \underbrace{(0.5z + 0.5)}_{\text{coin flip}}\underbrace{(0.5z + 0.5)}_{\text{coin flip}}\underbrace{(0.6z + 0.4)}_{\text{coin flip}}$$



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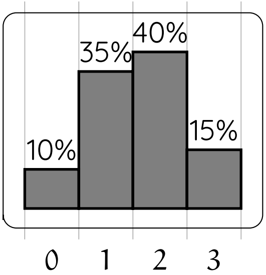
- When this happens, $\log(g(z))$ becomes concave over $\mathbb{R}_{\geq 0}$.



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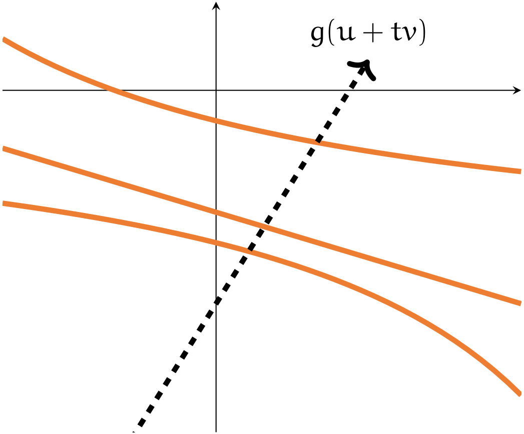
Main View in this Course

Properties of Polynomials $\leftrightarrow$ Efficiency of Algorithms

Example: Continuous Optimization

- Hyperbolic polynomial g: for some v and all u, $g(u+tv) \in \mathbb{R}[t]$ is real-rooted.



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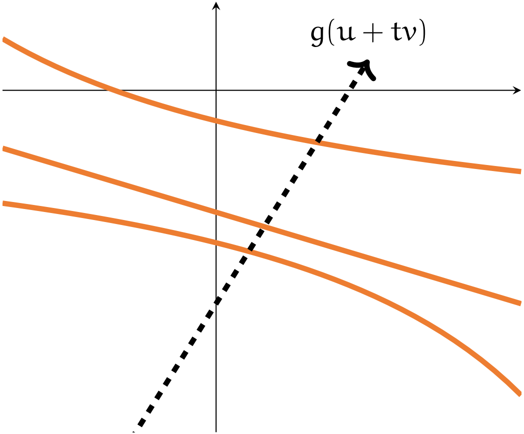

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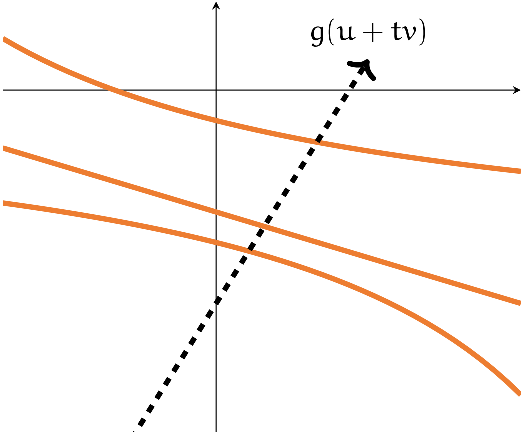

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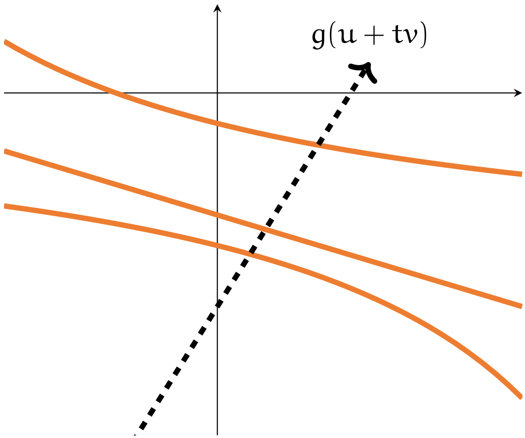
- Hyperbolicity $\implies$ region "above" roots convex.
- Barrier: $\log(g)$ is a concave function "above" the roots. Basis for hyperbolic programming.



Example: Random Walks

For weighted graph $G = (V, E)$, consider the degree-2 polynomial

$$g(z_{v_1}, \ldots, z_{v_n}) = \sum_{\{u,v\} \in E} w(u, v) z_u z_v$$

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Log-Concavity $\leftrightarrow$ Mixing

Random walk on G mixes if and only if $\log g(z_{v_1}^{\alpha}, \ldots, z_{v_n}^{\alpha})$ is concave in a neighborhood of $(1, \ldots, 1)$ for some $\alpha > 1/2$.

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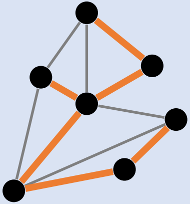
- Generalizes to hypergraphs; high-dimensional expanders.
- Efficient algorithms for sampling from combinatorial distributions.

Spanning Trees



uniformly at random

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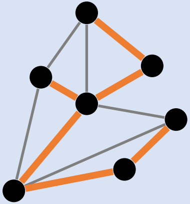

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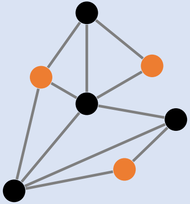
Stable Sets



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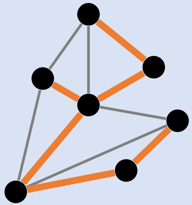

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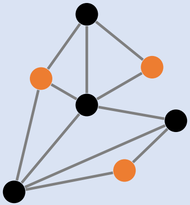

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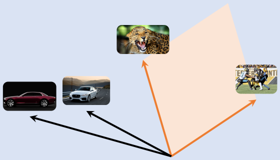
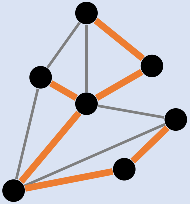
Volume Based



Prob $\propto$ volume

Spanning Trees



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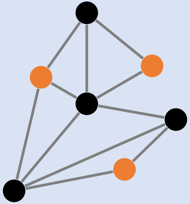

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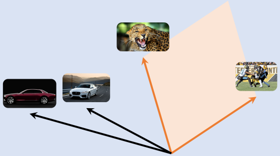

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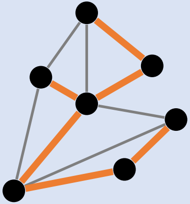

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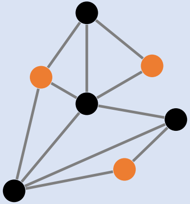

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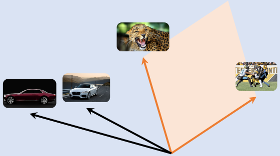

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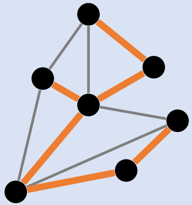

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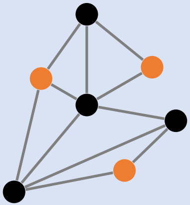

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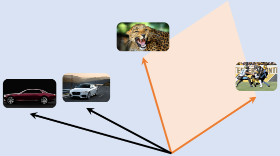

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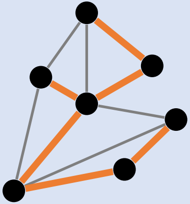

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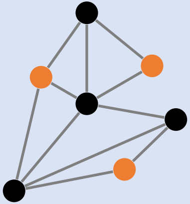

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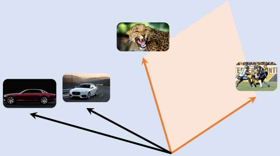

$\text{Prob} \propto \text{volume}^2$

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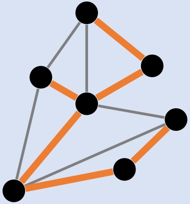

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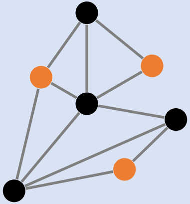

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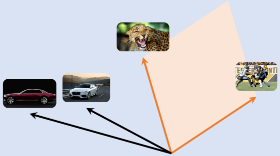

Prob $\propto$ volume^{10}

Example: Gross Substitutes [Kelsey-Crawford'82]

Suppose an agent wants to buy some subset of t-shirts with prices p_1, p_2, p_3:

$\$0$

$\$20 - p_1$

$\$10 - p_2$

$\$30 - p_1 - p_2$

$\$10 - p_3$

$\$30 - p_1 - p_3$

$\$10 - p_2 - p_3$

$\$30 - p_1 - p_2 - p_3$

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Suppose an agent wants to buy some subset of t-shirts with prices p_1, p_2, p_3:

- Rational Agent: Buy subset with most utility. Exponentially large table. ☹

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$\$30 - p_1 - p_2 - p_3$

Gross Substitutes

Locally rational agent finds the globally optimal subset.


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STANFORD

Example: Gross Substitutes + Discrete Choice

- Noisily Rational Agent [Nobel Prize: McFadden'00]: Buy S with probability:

$$\mathbb{P}[S] \propto e^{\text{utility}(S)}.$$

Exponentially large lookup table. ☹

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	$\$10 - p_2$
	$\$30 - p_1 - p_2$
	$\$10 - p_3$
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Theorem

"Random" additions, removals, replacements of one item at a time converge to the true distribution in $\sim O(n \log n)$ steps for gross substitutes.

$\$0$

$\$20 - p_1$

$\$10 - p_2$

$\$30 - p_1 - p_2$

$\$10 - p_3$

$\$30 - p_1 - p_3$

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Connection to Polynomials

$\$0$

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Connection to Polynomials

- The following multivariate polynomial captures the distribution

$$g(z_1, z_2, z_3) = e^0 + e^{20} z_1 + \cdots + e^{30} z_1 z_2 z_3.$$

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- This polynomial behaves like real-rooted univariate polynomials. In particular $\log g$ is concave over $\mathbb{R}^n_{\geqslant 0}$.
- Note: For univariate real-rooted polynomials

$$\log((0.5z + 0.5)(0.5z + 0.5)(0.6z + 0.4)) =$$

$$\log(0.5z + 0.5) + \log(0.5z + 0.5) + \log(0.6z + 0.4).$$

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Example: Permanent

Main Problem

Given $n \times n$ matrix M compute:

$$\text{per}(M) = \sum_{\text{permutation } \sigma} M_{1,\sigma(1)} M_{2,\sigma(2)} \cdots M_{n,\sigma(n)}.$$

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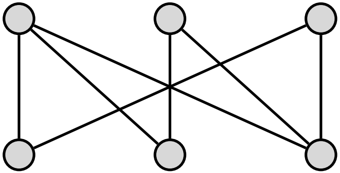

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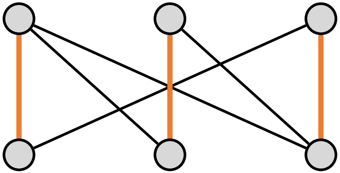

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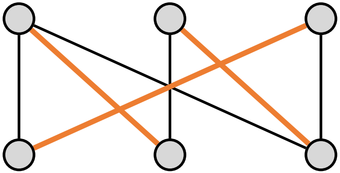

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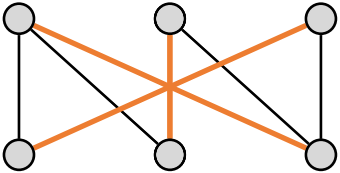

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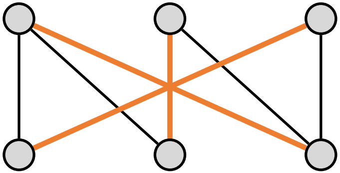

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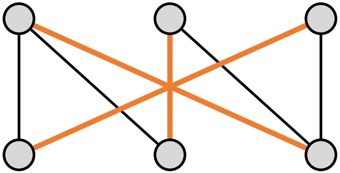

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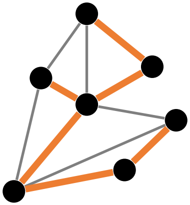
- #P-hard even for $0/1$ matrices [Valiant'79].
- When all of M is close to 1, Barvinok's method [on board ...]

Random Spanning Tree Degrees



Uniformly Random Spanning Tree

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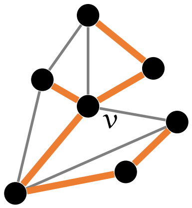


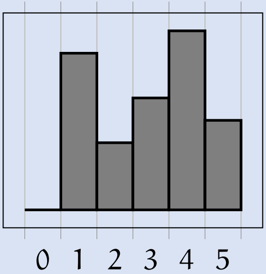
Uniformly Random Spanning Tree

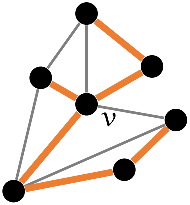
$\deg(v) = \#\{\text{edges adjacent to } v\}$

Random Spanning Tree Degrees

Shape of the Distribution

Can the dist. of $\deg(v)$ look like this?





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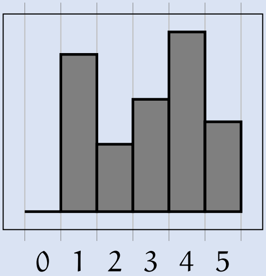
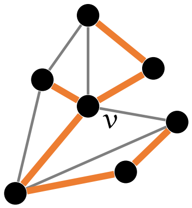


▷ No, it has to be unimodal.



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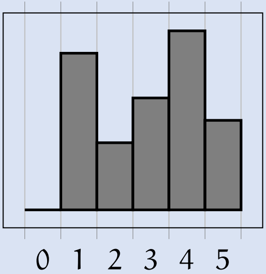

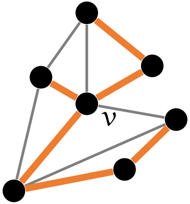

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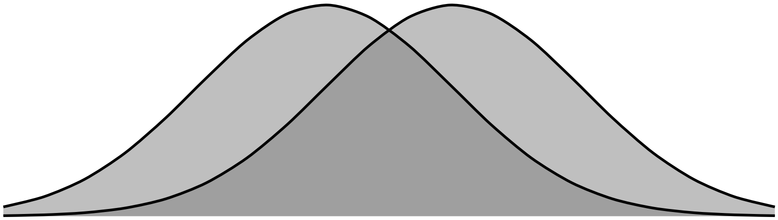
- No, it has to be unimodal.
- It should actually look like #heads in some number of independent (biased) coin flips. Has to be very concentrated around the mean value.

Test Your Intuition

- Two Gaussians. Are mixtures unimodal? Log-concave?



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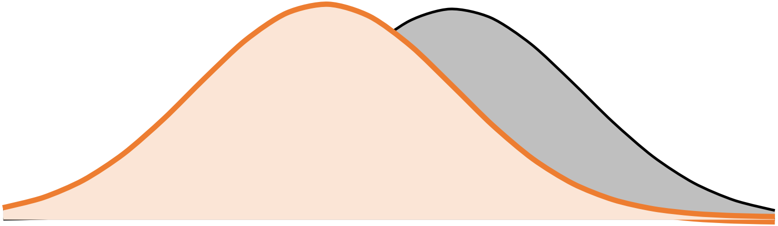

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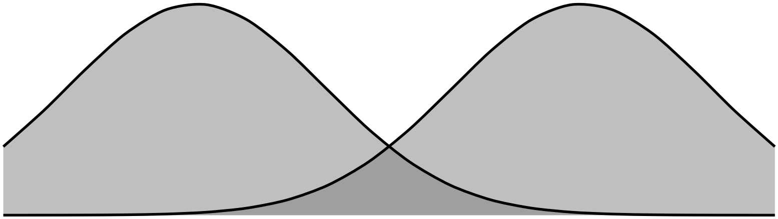

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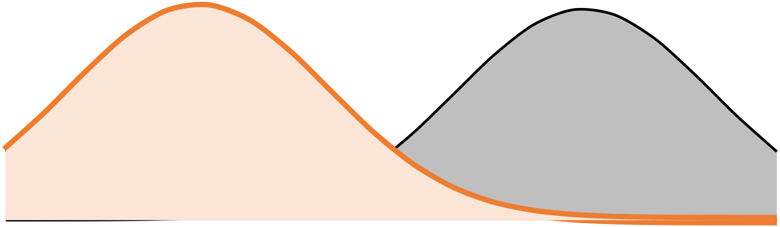

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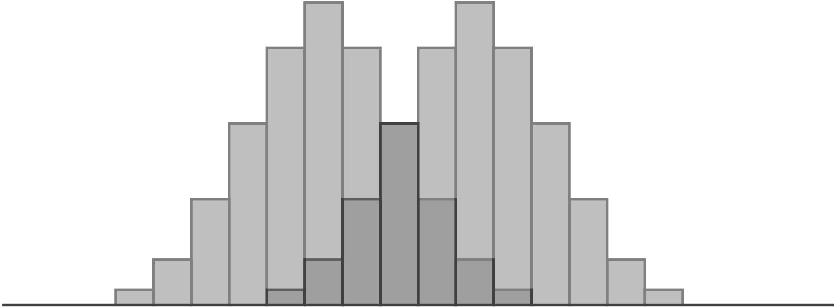

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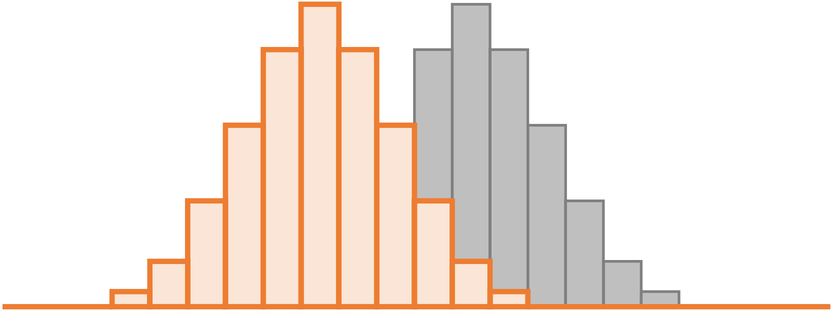

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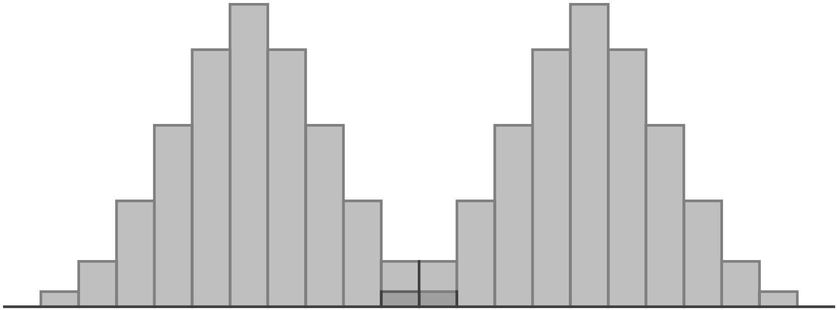

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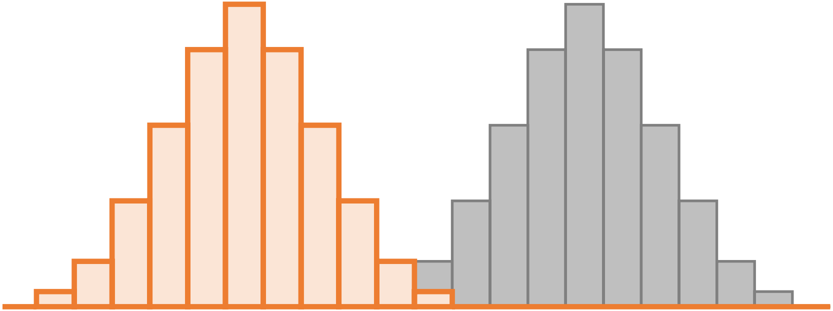

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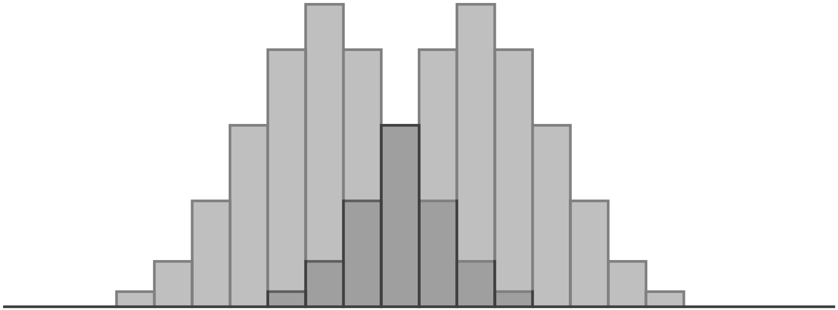
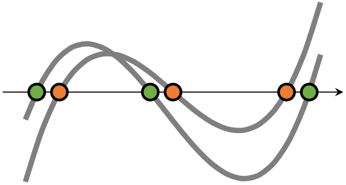


Mixtures of Polynomials

[Folklore, used by e.g., MSS'13]

If $\alpha g_1(z) + \beta g_2(z)$ is real-rooted for all $\alpha, \beta > 0$, then roots of g_1, g_2 must have common interlacing.



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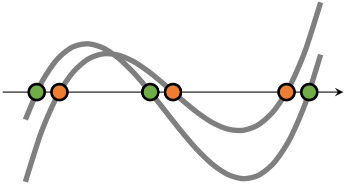

- Corollary: If mixtures of μ, ν are always generalized binomials, then

$$\mu = \text{Bernoulli}(p_1) + \cdots + \text{Bernoulli}(p_n),$$
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with $p_i, q_i \leqslant p_{i+1}, q_{i+1}$.

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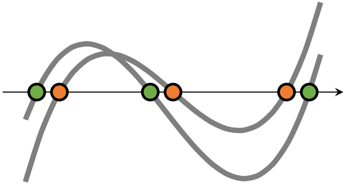

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- Corollary: The means of μ and ν can be off by $\leqslant 1$.

$$|(p_1 + \cdots + p_n) - (q_1 + \cdots + q_n)| \leqslant 1$$