# Notes: Interlacing Families

Nima Anari

`https://nimaanari.com/cs260-winter2020`

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## Recap

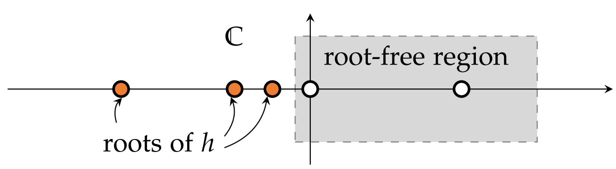
We studied roots of the matching polynomial for a graph $G = (V, E)$, which is defined by

$$m_G(z) = \sum_{M \text{ matching}} (-1)^{|M|} z^{|V|-2|M|}.$$

Our goal was to evaluate a similar polynomial that *counts* matchings

$$h(z) = \sum_{M \text{ matching}} z^{|M|} \propto \sqrt{z}^{|V|} m_G(i/\sqrt{z})$$

Because of this relation, for any root $\lambda$ of $m_G$ we have a root $-1/\lambda^2$ for $h$. We showed that $h$ has only nonpositive real roots, so to apply Barvinok's method and to approximately evaluate $h$ at $z = 1$ using derivatives of $h$ at $z = 0$, we simply needed to lower bound the distance of the closest root to zero.




We complemented this with an algorithm due to Patel and Regts [PR17] that can compute the coefficients of $z^0, z^1, \ldots, z^k$ in $h$ in time $\text{poly}(n)\Delta^k$, where $\Delta$ is the maximum degree in $G$. When $\Delta = O(1)$ and $k = O(\log n)$, this algorithm runs in polynomial time.

To show that $h$ has no roots close to 0, we switched to $m_G$ and showed that the maximum root of $m_G$ is bounded by $2\sqrt{\Delta - 1}$ [HL72]. We used the recursive definition of $m_G$ which for a vertex $v$ gives

$$m_G(z) = z \cdot m_{G-v}(z) - \sum_{u \sim v} m_{G-u-v}(z).$$

Our strategy was to fix a large enough value of $z$, and show that $m_G(z) > 0$; to show this we inductively showed the stronger statement: that for this fixed $z$, the ratio $m_G(z)/m_{G-v}(z) \geq \alpha$ for some $\alpha$ to be fixed later. For the induction to work, we needed

$$\frac{m_G(z)}{m_{G-v}(z)} = z - \sum_{u \sim v} \frac{m_{G-u-v}(z)}{m_{G-v}(z)} \geq z - \frac{\Delta}{\alpha}$$

to be at least $\alpha$. This was easy to satisfy when $z \geq 2\sqrt{\Delta}$, by letting $\alpha = \sqrt{\Delta}$. This proves that there are no roots larger than $2\sqrt{\Delta}$. To get the better bound of $2\sqrt{\Delta - 1}$, we noticed that except for the first step of the induction, we can always choose a vertex $v$ of degree $\leq \Delta - 1$. In the first step though, we don't need to prove the ratio $m_G(z)/m_{G-v}(z)$ is large, but rather that it is positive. Combining everything, we got

**Theorem 1** ([HL72])**.** *The matching polynomial $m_G(z)$ has no roots $z > 2\sqrt{\Delta - 1}$.*

## Interlacing Families

Our goal now is to use the derived bound on roots of $m_G$ to prove a result of Marcus, Spielman, and Srivastava [MSS13] on the existence of Ramanujan graphs. These are, roughly speaking, $\Delta$-regular graphs that are as good of an expander as they possibly can be. If $A_G$ denotes the adjacency matrix of a graph $G$, then the maximum eigenvalue of $A_G$ is

$$\lambda_1(A_G) = \Delta.$$

The best expanders have the second-largest eigenvalue, $\lambda_2$, bounded as far away as possible from $\lambda_1$. But there is a limit to this, known as the Alon-Bopanna bound

**Theorem 2** (Alon-Bopanna Bound)**.** *In any $\Delta$-regular graph $G$ with $n$ vertices*

$$\lambda_2(A_G) \geq 2\sqrt{\Delta - 1} - o_n(1).$$

For large enough $n$, we can only hope for $\lambda_2$ to be bounded by $2\sqrt{\Delta - 1}$. Roughly speaking, Ramanujan graphs are the ones that achieve this.

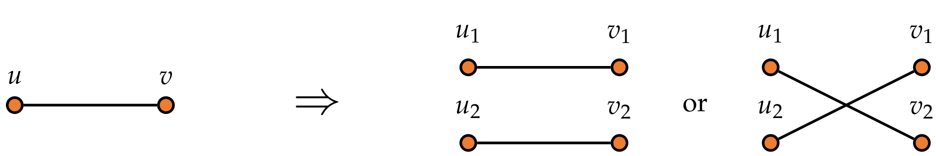
*Remark* 3. To be precise, in expander graphs we sometimes also care about the magnitude of negative eigenvalues, and so we might care that both $\lambda_2(A_G)$ and $|\lambda_n(A_G)|$ are bounded by $2\sqrt{\Delta - 1}$; this is the most prevalent definition of a Ramanujan graph. Here in this note, we ignore $\lambda_n$, and try to only bound $\lambda_2$. In particular for just this note, Ramanujan means $\lambda_2 \leq 2\sqrt{\Delta - 1}$. In the end, the construction will yield bipartite graphs, and these graphs have eigenvalues symmetric about zero. So this puts the same bound of $2\sqrt{\Delta - 1}$ on all of $|\lambda_2|, |\lambda_3|, \ldots, |\lambda_{n-1}|$. Note that in bipartite $\Delta$-regular graphs, necessarily $|\lambda_1| = |\lambda_n| = \Delta$, so in some sense these bipartite graphs are the best possible bipartite expanders.

We will prove the following.

**Theorem 4** (Marcus, Spielman, and Srivastava [MSS13])**.** *For any $\Delta$, there are an infinite number of $\Delta$-regular bipartite graphs that satisfy the Ramanujan bound of $\lambda_2(A_G) \leq 2\sqrt{\Delta - 1}$.*

The proof uses the concept of a 2-lift, an operation with roots in algebraic topology. For a graph $G = (V, E)$, a 2-lift is another graph $H$ on twice as many vertices. For each vertex $v$ of $G$ we make two copies $v_1, v_2$ in $H$. For each edge $(u, v)$, we introduce two edges in $H$:

- either $(u_1, v_1)$ and $(u_2, v_2)$, or
- $(u_1, v_2)$ and $(u_2, v_1)$.




Note that a separate choice is to be made for each edge of $G$, and any collection of these choices results in one 2-lift $H$. The adjacency matrix of $H$ has a block form

$$A_H = \begin{bmatrix} B & C \\ C & B \end{bmatrix}$$

where $B$ indicates the edges for which we made a choice of $(u_1, v_1), (u_2, v_2)$, and $C$ indicates the other edges. In particular, $B + C = A_G$. To analyze the eigenvalues of this matrix, it is more convenient to transform it first:

$$\frac{1}{2}\begin{bmatrix} I & I \\ -I & I \end{bmatrix} \cdot \begin{bmatrix} B & C \\ C & B \end{bmatrix} \cdot \begin{bmatrix} I & -I \\ I & I \end{bmatrix} = \begin{bmatrix} B+C & 0 \\ 0 & B-C \end{bmatrix} = \begin{bmatrix} A_G & 0 \\ 0 & B-C \end{bmatrix}$$

The above shows that the matrix $A_H$ is similar to the matrix on the r.h.s. This means that their eigenvalues are the same, and conveniently the r.h.s. has a block form. So the eigenvalues of $A_H$ are the union of eigenvalues of $A_G$ and the matrix $B - C$. The matrix $B - C$ is a signed version of $A_G$, where we replace some of the 1s by $-1$.

**Definition 5.** $D$ is a signing of the adjacency matrix $A_G$, if it is a symmetric matrix that can be obtained by replacing some of the 1s with $-1$ in $A_G$.

Signings of the adjacency matrix are exactly the set of matrices we can get as $B - C$ in the 2-lift. So, we obtain the following convenient statement:

**Proposition 6.** *If $G$ is a $\Delta$-regular Ramanujan graph, i.e., $\lambda_2(A_G) \leq 2\sqrt{\Delta - 1}$, and $D$ is a signing of its adjacency matrix with $\lambda_1(D) \leq 2\sqrt{\Delta - 1}$, then the 2-lift corresponding to $D$ is still a $\Delta$-regular Ramanujan graph.*

So in order to grow an infinite family of $\Delta$-regular Ramanujan graphs, it is enough to start with one, and each time find a "good" signing of its adjacency matrix. This is what we will prove next

**Theorem 7** (Marcus, Spielman, and Srivastava [MSS13])**.** *For every $\Delta$-regular graph $G$, there is a signing $D$ of the adjacency matrix such that $\lambda_1(D) \leq 2\sqrt{\Delta - 1}$.*

Over the years there have been many proposals on how to construct these signings. Signing each edge uniformly and independently at random is a pretty good choice, and one might be tempted to analyze $\mathbb{E}[\lambda_1(D)]$ over these random signings $D$. This quantity however can become close to $\Delta \gg 2\sqrt{\Delta - 1}$ as $n$ grows. Think of a graph that is a disjoint union of many constant-sized $\Delta$-regular graphs. In a random signing, at least one component will be signed positive with high probability, and this will produce an eigenvalue of $\lambda_1(D) = \Delta$. So with very high probability we will have $\lambda_1(D) = \Delta$.

Even though on average the maximum root does not follow the right bound, Marcus, Spielman, and Srivastava [MSS13] showed that the average polynomial does! For each signing, we can look at the characteristic polynomial $\chi_D(z) = \det(zI - D)$, whose maximum root is $\lambda_1(D)$. The unexpected revelation is that $\mathbb{E}_D[\chi_D(z)]$ ends up being a polynomial with real roots whose maximum root is $\leq 2\sqrt{\Delta - 1}$. Moreover, this *average polynomial*'s roots end up really being in the middle of the roots of $\chi_D$s, that is

$$\exists \text{ signing } D : \text{max-root}(\chi_D) \leq \text{max-root}(\mathbb{E}_D[\chi_D])$$

We will prove these in Lemmas 8 and 9. Together these lemmas imply that if we start with any regular Ramanujan graph, we can grow an infinite family of them. It is an easy exercise to check that the complete bipartite graph $K_{\Delta,\Delta}$ is a Ramanujan graph (its second eigenvalue is 0); so we can always start growing the Ramanujan family from it.

We will first prove the bound on the maximum root of the average polynomial, by showing that the average polynomial is actually the matching polynomial of the graph.

**Lemma 8.** *For every graph $G = (V, E)$, the average characteristic polynomial over signings is the matching polynomial:*

$$\mathbb{E}_D[\det(zI - D)] = m_G(z).$$

*Proof.* We can write the determinant using a signed sum of permuted diagonals:

$$\mathbb{E}_D[\det(zI - D)] = \mathbb{E}_D\left[\sum_{\text{permutation } \pi} (-1)^{\text{sign}(\pi)} \prod_{i=1}^{n} (zI - D)_{i,\pi(i)}\right]$$

Note that for each $\pi$, if $\pi(\pi(i)) \neq i$ for any $i \in [n]$, then the expected value of the term corresponding to $\pi$ is 0. This is because when averaged over random signings, the entry at $(i, \pi(i))$ is either identically 0 or takes $\pm 1$ values uniformly at random. This entry is independent of every other entry, except for the one at $(\pi(i), i)$. So, as long as $(\pi(i), i)$ does not appear in the product, the average cancels out.

So we just have to keep the terms corresponding to permutations $\pi$ where $\pi \circ \pi = \mathrm{id}$. Each such permutation corresponds to a matching $M$; the indices $i$ where $\pi(i) = i$ are the monomers or unmatched vertices, and all other indices get paired up. If any of the edges in $M$ are not part of the graph $G = (V, E)$, the corresponding entry in $zI - D$ is zero, and the term cancels out. So we only need to consider permutations corresponding to matchings $M$ of the graph $G$. It is an easy exercise to see that $\mathrm{sign}(\pi) = |M|$. Also note that regardless of the signing $D$, for every such $\pi$ we have

$$\prod_{i=1}^{n} (zI - D)_{i,\pi(i)} = z^{|V|-2|M|}.$$

We get one factor of $z$ for every index $i$ where $\pi(i) = i$, and for all other indices the signs cancel each other. Putting everything together, we get that

$$\mathbb{E}_D[\det(zI - D)] = \sum_{\text{matching } M} (-1)^{|M|} z^{|V|-2|M|} = m_G(z).$$

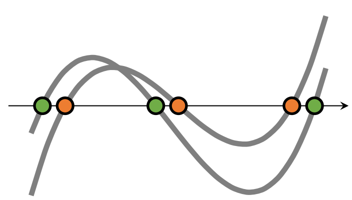
□

**Lemma 9.** *There exists a signing $D$ with*

$$\text{max-root}(\chi_D) \leq \text{max-root}(\mathbb{E}_D[\chi_D])$$

*Proof.* A priori, an average of a number of polynomials does not even need to have real roots, let alone roots that are sandwiched by the original polynomials. Consider the example of $(z-1)^2 = z^2 - 2z + 1$ and $(z+1)^2 = z^2 + 2z + 1$ whose average is $z^2 + 1$, a polynomial with imaginary roots.

However, we saw in the beginning of the course, that if the roots of two univariate polynomials interlace, then any positive combination of them, i.e., any weighted average of them, becomes real-rooted and vice versa; moreover the roots of the average will be sandwiched between the roots of the two polynomials.



Our strategy here is to prove that for *biased signings $D$*, the average characteristic polynomial $\chi_D$ is real-rooted, no matter the bias. We will then show how this implies sandwiching of the roots.

If $m$ is the number of edges, we define for a vector $p = (p_1, \ldots, p_m) \in [0,1]^m$, a $p$-biased signing $D$, obtained by signing each edge $e$, $+1$ with probability $p_e$ and $-1$ with probability $1 - p_e$, independently of all other edges. Let $\mu_p$ denote the distribution of $p$-biased signings. We will prove that

$$g_p(z) := \mathbb{E}_{D \sim \mu_p}[\chi_D] = \mathbb{E}_{D \sim \mu_p}[\det(zI - D)]$$

is a real-rooted polynomial. We defer the proof to Lemma 10. Let us see how the real-rootedness of $g_p$ implies sandwiching of the roots.

We can expand $g_p$ in terms of its first variable:

$$g_p(z) = p_1 \cdot g_{(0,p_2,\dots,p_m)}(z) + (1-p_1) \cdot g_{(1,p_2,\dots,p_m)}.$$

Note that for any choice of $p_1$ the above is a real-rooted polynomial by Lemma 10. So we have two polynomials, all of whose convex combinations are real-rooted. Therefore, their roots must interlace and the roots of their average must be sandwiched between the roots of the two polynomials. This means that either

$$\text{max-root}(g_{(0,p_2,\dots,p_m)}) \le \text{max-root}(g_p) \le \text{max-root}(g_{(1,p_2,\dots,p_m)})$$

or

$$\text{max-root}(g_{(1,p_2,\dots,p_m)}) \le \text{max-root}(g_p) \le \text{max-root}(g_{(0,p_2,\dots,p_m)}).$$

In either case, we find a polynomial with a not-larger max-root. Now we can apply the same trick to the second coordinate, and so on. In the end we find some choice $(q_1, \dots, q_m) \in \{0,1\}^m$ such that

$$\text{max-root}(g_{(q_1,\dots,q_m)}) \le \text{max-root}(g_p).$$

But $\mu_q$ is a deterministic distribution on a single signing. So we have found a signing whose max-root is at most the initial max-root. $\square$

**Lemma 10.** *For any choice $(p_1, \dots, p_m) \in [0,1]^m$, the biased average polynomial*

$$\mathbb{E}_{D\sim\mu_p}[\det(zI - D)]$$

*is real-rooted.*

*Proof.* For every edge $e$, let us define two vectors $u_e, v_e \in \mathbb{R}^n$. Both vectors have 0 as their every entry, except for the endpoints of the edge $e$; for the endpoints, $v_e$ has $+1$ at both places, and $u_e$ has one $+1$ and one $-1$ (arbitrarily assigned). With these choices, it is easy to express any signing $D$:

$$D = \sum_{e \text{ signed } +1} u_e u_e^\intercal + \sum_{e \text{ signed } -1} v_e v_e^\intercal - \Delta \cdot I$$

Note that $\Delta \cdot I$ is subtracted to cancel the diagonals contributed by $u_e u_e^\intercal$ and $v_e v_e^\intercal$ (both contributed $+1$ to the diagonal entries of the endpoints of $e$). This representation allows us to start with a real-stable polynomial and obtain $\mathbb{E}_{D\sim\mu_p}[\chi_D]$ by a sequence of operations that preserve real-stability.

First, note that the polynomial

$$\det(zI + x_1 u_1 u_1^\intercal + y_1 v_1 v_1^\intercal + \cdots + x_m u_m u_m^\intercal + y_m v_m v_m^\intercal)$$

is real-stable in $z, x_1, \dots, x_m, y_1, \dots, y_m$. This is because the matrix in front of each variable is positive semidefinite. By performing a shift operation $z \mapsto z + \Delta$, we preserve real-stability and get

$$h := \det((z+\Delta)I + x_1 u_1 u_1^\intercal + y_1 v_1 v_1^\intercal + \cdots + x_m u_m u_m^\intercal + y_m v_m v_m^\intercal).$$

We will prove that

$$\mathbb{E}_{D\sim\mu_p}[\chi_D] = \left(\prod_e (1 - p_e \partial_{x_e} - (1-p_e)\partial_{y_e})\right) h \Bigg|_{x=y=0}.$$

Each of the operations $1 - p_e \partial_{x_e} - (1-p_e)\partial_{y_e}$ preserves real-stability, as does setting the variables in the end to 0. So by proving this identity, we will finish the proof of real-rootedness.

Let us simplify a bit, and consider what these operations do, bivariate polynomials. Our polynomial is multiaffine in the $x$ and $y$ variables, because the matrices in front of these variables are rank 1. So, consider a multiaffine bivariate polynomial $g(x, y) = a + bx + cy + dxy$. Then

$$(1 - p\partial_x - (1-p)\partial_y)g\big|_{x=y=0} = a - pb - (1-p)c$$

But this quantity is the same as

$$p(a-b) + (1-p)(a-c) = p \cdot g(-1, 0) + (1-p) \cdot g(0, -1).$$

So we can interpret this operation on $g$, as plugging in $(-1, 0)$ with probability $p$ and $(0, -1)$ with probability $1 - p$ for the variables $x, y$, and then taking the average.

This probabilistic interpretation carries out to the multivariate setting as well. Because $h$ is multiaffine in the $x_e$ and $y_e$ variables, we get that

$$\left(\prod_e (1 - p_e\partial_{x_e} - (1-p_e)\partial_{y_e})\right)h\Bigg|_{x=y=0} = \mathbb{E}_{x,y}[h(z, x_1, y_1, \ldots, x_m, y_m)],$$

where in the expectation, each $(x_e, y_e)$, independently of others, takes the value $(-1, 0)$ and $(0, -1)$ with probabilities $p_e$ and $1 - p_e$. But note that for any of these choices we have $h(z, x_1, y_1, \ldots, x_m, y_m) = \det(zI - (\sum_{e:x_e=-1} u_e u_e^\intercal + \sum_{e:y_e=-1} v_e v_e^\intercal - \Delta \cdot I)) = \chi_D$ where $D$ is the appropriate signing encoding the $(-1, 0)/(0, -1)$ choices. $\square$

# References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