# Problem Set 2

CS 260: Geometry of Polynomials in Algorithm Design

https://nimaanari.com/cs260-winter2020/

## 1 Approximate Negative Correlation

Let $D_{k\to l} \in \mathbb{R}^{\binom{[n]}{k}\times\binom{[n]}{l}}$ be the first half (down part) of the random walk between sets of size $k$ and sets of size $l$, defined by

$$D_{k\to l}(S,T) = \begin{cases} 1/\binom{k}{l} & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

For any distribution $\nu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$, applying $D_{k\to l}$ gives us a distribution $\nu D_{k\to l} : \binom{[n]}{l} \to \mathbb{R}_{\geq 0}$ defined by the vector-matrix product

$$\nu D_{k\to l}(T) = \sum_{S\in\binom{[n]}{k}} \nu(S) D_{k\to l}(S,T) = \frac{\sum_{S\supseteq T} \nu(S)}{\binom{k}{l}}.$$

In class we showed that if $\nu, \mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ are distributions where $\nu$ is arbitrary and $\mu$ has a log-concave polynomial $g_\mu(z_1, \ldots, z_n) = \sum_S \mu(S) \prod_{i\in S} z_i$, then $D_{k\to l}$ shrinks their KL-divergence by a factor of $l/k$. In other words

$$D_{\mathrm{KL}}(\nu D_{k\to l} \parallel \mu D_{k\to l}) \leq \frac{l}{k} D_{\mathrm{KL}}(\nu \parallel \mu).$$

As a reminder, the KL-divergence is defined as the $f$-divergence with $f(x) = x \log x$:

$$D_f(\nu \parallel \mu) := \mathbb{E}_{S\sim\mu}[f(\nu(S)/\mu(S))] - f(\mathbb{E}_{S\sim\mu}[\nu(S)/\mu(S)]).$$

In the following, let $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ be a distribution with a log-concave polynomial.

1. Let $T \in \binom{[n]}{k}$ be a fixed set and let $\nu = \mathbb{1}_T$ be the distribution whose entire probability mass is on $T$. Use the shrinkage of the KL-divergence for the operator $D_{k\to 1}$ to prove the following inequality relating $\mu(T)$ and marginals of $\mu$:

   $$\mathbb{P}_{S\sim\mu}[S = T] \leq \prod_{i\in T} \mathbb{P}_{S\sim\mu}[i \in S].$$

2. Now let $l \in \{0, 1, \ldots, k\}$ be some integer and let $T \in \binom{[n]}{l}$ be a fixed set. Define $\nu = \mathbb{1}_T$ to be the distribution whose entire probability mass is on the set $T$. Let $\mu' = \mu D_{k\to l}$. Note that $\nu$ and $\mu'$ are defined on the same ground set, and that $\mu'$ has a log-concave polynomial. What is the value of $\mu'(T)$? How about the marginals of $\mu'$? Use the results of the previous part to obtain

   $$\mathbb{P}_{S\sim\mu}[T \subseteq S] \leq c(k,l) \prod_{i\in T} \mathbb{P}_{S\sim\mu}[i \in S],$$

   for some function $c(k,l)$.

3. Show that we can take $c(k,l) \leq l^l/l!$. Conclude that $c(k,2) \leq 2$ and $c(k,l) \leq e^l$.