Problem Set 1

CS 260: Geometry of Polynomials in Algorithm Design

https://nimaanari.com/cs260-winter2020/

1 Real Stability vs. Log-Concavity

Let $g(z_1, z_2) = a + bz_1 + cz_2 + dz_1z_2$ be a multi-affine bivariate polynomial with $a, b, c, d \in \mathbb{R}_{\geq 0}$. Prove that

1) g is real stable if and only if $ad \leq bc$.
2) $\log g$ is concave over $\mathbb{R}^2_{\geq 0}$ if and only if $ad \leq 2bc$.

2 Approximating Higher Coefficients

Let $g(z_1, \dots, z_n)$ be a real stable polynomial with nonnegative coefficients, and let $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$. Let

$$\alpha := \inf_{z_1, \dots, z_n > 0} \frac{g(z_1, \dots, z_n)}{z_1^{k_1} \cdots z_n^{k_n}}$$

and let β be the coefficient of the $z_1^{k_1} \cdots z_n^{k_n}$ term in g. Show α approximates β by following these steps:

1) Prove that $\alpha \geq \beta$.
2) Suppose $n = 1$, and $g(z) = e^{cz}$ for some $c \in \mathbb{R}_{\geq 0}$. Note that g is not a polynomial, but you can think of it in terms of its Taylor series

$$g(z) = 1 + cz + \frac{c^2}{2} z^2 + \cdots + \frac{c^k}{k!} z^k + \dots$$

   The coefficient of z^k in the above expression is $c^k/k!$. What is $\inf_{z>0} g(z)/z^k$? Show that the ratio between these two quantities (α/β) is only a function of k and not c. Let this quantity be $f(k)$.
3) [Bonus, no need to turn in solution to this part] Show that $f(k) = O(\sqrt{k})$.
4) Now let $n = 1$ and $g(z)$ a real-rooted polynomial

$$g(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_d z^d.$$

   We showed in class that $\{c_i/\binom{d}{i}\}_{i=0}^d$ is a log-concave sequence. Show that this implies $\{\partial_z^i g(0)\}_{i=0}^d$ is a log-concave sequence as well.
5) Let $r, s \in \mathbb{R}_{\geq 0}$ and define $h(z) = re^{sz}$. What quality does the sequence $\{\partial_z^i h(0)\}_{i=0}^\infty$ have? Show that for some choice of r and s we will have $\partial_z^i h(0) \geq \partial_z^i g(0)$ for all i and that equality happens for $i = k$, i.e., $\partial_z^k h(0) = \partial_z^k g(0)$.
6) Conclude that for $n = 1$, $f(k) \cdot \beta \geq \alpha$.
7) Prove using induction that for general n

$$f(k_1) f(k_2) \cdots f(k_n) \beta \geq \alpha.$$

3 Monomer-Dimer Correlations

Let $G = (V, E)$ be a graph with an edge weight function $w : E \to \mathbb{R}_{\geq 0}$ and vertex weight function $\lambda : V \to \mathbb{R}_{\geq 0}$. For a matching $M \subseteq E$, we call the vertices v not adjacent to M the monomers of M.

Let μ be the monomer distribution defined on subsets of V, where

$$\mu(S) \propto \sum_{M \text{ matching with monomer set } S} \prod_{v \in S} \lambda(v) \prod_{e \in M} w(e).$$

Let g be the following polynomial encoding μ in its coefficients.

$$g(z_1, \ldots, z_n) = \sum_S \mu(S) \prod_{i \in S} z_i.$$

We showed in class that for $z_1, \ldots, z_n \in \mathbb{C}$ with $\mathrm{Re}(z_1), \ldots, \mathrm{Re}(z_n) > 0$ we have $g(z_1, \ldots, z_n) \neq 0$.

1) Let $T \subseteq V$ and define the random variable X by sampling S according to μ and computing

$$X := |V| + |S \cap T| - |S \cap (V - T)|.$$

Show that X takes only even nonnegative integer values.

2) Let $h(z)$ be the following polynomial

$$h(z) := \mathbb{P}[X = 0]z^0 + \mathbb{P}[X = 2]z^2 + \cdots + \mathbb{P}[X = 2k]z^{2k} + \ldots.$$

Express h in terms of g. Show that h has roots only on the imaginary axis of the complex plane.

3) Show that $h(\sqrt{z})$ is a real-rooted polynomial in z. Conclude that $X/2$ is distributed as a sum of independent Bernoulli random variables.

4) Let $v \in V$, and look at the conditional distribution of $X/2$ conditioned on $v \in S$. Show that this is also a sum of independent Bernoullis. [Hint: vary the weight $\lambda(v)$]

5) Show that any mixture of the distribution of $X/2$ and the conditional distribution of $X/2$ (conditioned on $v \in S$) is also a sum of independent Bernoullis. Conclude that

$$|\mathbb{E}[X] - \mathbb{E}[X \mid v \in S]| \leq 2.$$

6) Let $C \in \mathbb{R}^{n \times n}$ be a matrix defined by

$$C_{vu} = \mathbb{P}[u \in S \mid v \in S] - \mathbb{P}[u \in S].$$

Show that the ℓ_1 norm of the v-th row of C is exactly the quantity $|\mathbb{E}[X] - \mathbb{E}[X \mid v \in S]|$ for some choice of T. Conclude that the ℓ_1 norm of rows of C are bounded by 2.