Log-Concave Polynomials III: Mason's Ultra-Log-Concavity Conjecture for Independent Sets of Matroids

Nima Anari[1], Kuikui Liu[2], Shayan Oveis Gharan[2], and Cynthia Vinzant[3]

[1]Stanford University, anari@cs.stanford.edu
[2]University of Washington, liukui17@cs.washington.edu, shayan@cs.washington.edu
[3]North Carolina State University, clvinzan@ncsu.edu

November 5, 2018

Abstract

We give a self-contained proof of the strongest version of Mason's conjecture, namely that for any matroid the sequence of the number of independent sets of given sizes is ultra log-concave. To do this, we introduce a class of polynomials, called completely log-concave polynomials, whose bivariate restrictions have ultra log-concave coefficients. At the heart of our proof we show that for any matroid, the homogenization of the generating polynomial of its independent sets is completely log-concave.

1 Introduction

Matroids are combinatorial structures that model various types of independence, such as linear independence of vectors in a linear space or algebraic independence of elements in a field extension. For an inspiring recent survey, see [Ard18]. There have been several recent breakthroughs proving inequalities on sequences of numbers associated to matroids. While the proofs in this paper are self-contained, we build off several of these ideas to study the following conjecture of Mason [Mas72].

Conjecture 1.1 (Mason's Conjecture)**.** *For an n-element matroid M with $\mathcal{I}_k$ independent sets of size k,*

i) $\mathcal{I}_k^2 \geq \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$ *(log-concavity),*

ii) $\mathcal{I}_k^2 \geq \left(1 + \frac{1}{k}\right) \cdot \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$*,*

iii) $\mathcal{I}_k^2 \geq \left(1 + \frac{1}{k}\right) \cdot \left(1 + \frac{1}{n-k}\right) \cdot \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$ *(ultra log-concavity).*

Note that (i) to (iii) are written in increasing strength. Adiprasito, Huh, and Katz [AHK18] proved (i) using techniques from Hodge theory and algebraic geometry. Building on this, Huh, Schröter, and Wang [HSW18] proved (ii). Prior to our work, (iii) was only proven to hold when $n \leq 11$ or $k \leq 5$ [KN11]. We refer to [Sey75; Dow80; Mah85; Zha85; HK12; HS89; Len13] for other partial results on Mason's conjecture. Here, we give a self-contained proof of (iii).

Theorem 1.2. *For a matroid M on n elements with $\mathcal{I}_k$ independent sets of size k, the sequence $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$ is ultra log-concave. That is, for $1 < k < n$,*

$$\left(\frac{\mathcal{I}_k}{\binom{n}{k}}\right)^2 \quad \geq \quad \frac{\mathcal{I}_{k-1}}{\binom{n}{k-1}} \cdot \frac{\mathcal{I}_{k+1}}{\binom{n}{k+1}}.$$

We prove Theorem 1.2 in Section 5. The main tool we use will be polynomials that are log-concave as functions on the positive orthant. For $i \in [n]$, let ∂_i or ∂_{z_i} denote the partial derivative operator that maps a polynomial f to its partial derivative with respect to z_i. For a vector $v \in \mathbb{R}^n$, we let D_v denote the directional derivative operator in direction v,

$$D_v = \sum_{i=1}^{n} v_i \partial_i.$$

We call a polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ log-concave over $\mathbb{R}^n_{\geq 0}$ if f is nonnegative and log-concave as a function over $\mathbb{R}^n_{\geq 0}$, or in other words if for every $u, v \in \mathbb{R}^n_{\geq 0}$ and $\lambda \in [0,1]$, we have $f(u), f(v) \geq 0$ and

$$f(\lambda u + (1-\lambda)v) \geq f(u)^{\lambda} \cdot f(v)^{1-\lambda}.$$

Note that the zero polynomial is also log-concave. If $f(v)$ is positive for some $v \in \mathbb{R}^n_{\geq 0}$, then we call f log-concave at $z = v$ if the Hessian of its log at v is negative semidefinite. It is easy to see from the definition that for any fixed d and n, the set of polynomials of degree at most d in n variables that are log-concave on $\mathbb{R}^n_{\geq 0}$ is closed in the Euclidean topology on $\mathbb{R}[x_1, \ldots, x_n]_{\leq d}$. Also, a nonzero polynomial is log-concave over $\mathbb{R}^n_{\geq 0}$ if and only if it is log-concave at every point of $\mathbb{R}^n_{>0}$.

Definition 1.3. A polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is **completely log-concave** if for every set of nonnegative vectors $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0}$, the polynomial $D_{v_1} \ldots D_{v_k} f$ is nonnegative and log-concave over $\mathbb{R}^n_{\geq 0}$.

Completely log-concave polynomials were introduced in [AOV18] based on similar notions of strongly log-concave and Alexandrov-Fenchel polynomials first studied in [Gur09]. In this paper, we prove the properties of complete log-concavity necessary for Theorem 1.2 and defer a more detailed treatment of completely log-concave polynomials to a future article.

The main ingredient of the proof of Theorem 1.2 is to show that the homogenization of the generating polynomial of all independent sets of M is completely log-concave, namely that the polynomial

$$g_M(y, z_1, \ldots, z_n) = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} z_i$$

is completely log-concave. Then, we use this to show that the bivariate restriction $f_M(y,z) = \sum_{k=0}^{r} \mathcal{I}_k y^{n-k} z^k$ is completely log-concave. Finally, we derive Theorem 1.2 from the latter fact based on an observation of Gurvits [Gur09] on the coefficients of completely log-concave polynomials.

1.1 Independent work

In a related upcoming work, Brändén and Huh have independently developed methods that overlap with our work. In particular they also prove the strongest version of Mason's conjecture.

1.2 Spectral negative dependence

It is well-known that the uniform distribution over all spanning trees of a graph is negatively correlated and more generally negatively associated, see [Pem00] for background. This fact more generally extends to regular matroids. Prior to our work many researchers tried to approach Mason's conjecture through the lens of negative correlation [SW75; Wag08; BBL09; KN10; KN11]. However, for many matroids the uniform distribution on bases is not negatively correlated and furthermore, negative correlation does not necessarily imply log-concavity of its rank sequences [Wag08].

Consider the polynomial $p_M = \sum_B \prod_{i \in B} z_i$, where the sum is over all bases of the matroid M. Then the negative correlation property is equivalent to all off-diagonal entries of the Hessian of $\log p_M$ being non-positive when evaluated at the all-ones vector $\mathbb{1} = (1, \ldots, 1)$, i.e.

$$(\nabla^2 \log p_M(\mathbb{1}))_{i,j} = p_M(\mathbb{1}) \cdot \partial_i \partial_j p(\mathbb{1}) - \partial_i p_M(\mathbb{1}) \cdot \partial_j p_M(\mathbb{1}) \leq 0,$$

for all $1 \leq i, j \leq n$, $i \neq j$. This inequality holds for regular matroids but not necessarily for even linear matroids.

In [AOV18] it was observed that for any matroid M, the polynomial p_M is completely log-concave. This means that even though $\nabla^2 \log p_M(\mathbb{1})$ can have positive entries, all of its eigenvalues, and eigenvalues of Hessian of the log of all partials of p_M, are non-positive. We call this property, *spectral negative dependence*. In this paper, we show that for any matroid, the homogenization of the generating polynomial of all independent sets, namely g_M also satisfies spectral negative dependence. Furthermore, spectral negative dependence is enough to prove the strong form of log-concavity of rank sequences as conjectured by Mason.

Acknowledgements. Part of this work was started while the first and last authors were visiting the Simons Institute for the Theory of Computing. It was partially supported by the DIMACS/Simons Collaboration on Bridging Continuous and Discrete Optimization through NSF grant CCF-1740425. Shayan Oveis Gharan and Kuikui Liu are supported by the NSF grant CCF-1552097 and ONR-YIP grant N00014-17-1-2429. Cynthia Vinzant was partially supported by the National Science Foundation grant DMS-1620014.

2 Preliminaries

A polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is homogeneous of degree d if every monomial of f has degree d, or equivalently $f(\lambda \cdot z_1, \ldots, \lambda \cdot z_n) = \lambda^d f(z_1, \ldots, z_n)$ for all $\lambda \in \mathbb{R}$. We will use ∇f to denote the gradient of f and $\nabla^2 f$ to denote its Hessian matrix.

We use $[n]$ to refer to $\{1, \ldots, n\}$. When n is clear from context, for a set $S \subseteq [n]$, we let $\mathbb{1}_S \in \mathbb{R}^n$ denote the indicator vector of S. For variables $z_1, \ldots, z_n$ and $S \subseteq [n]$, we let z^S denote the monomial $\prod_{i \in S} z_i$. Similarly, for an integer vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ or a subset $S \subseteq [n]$, we denote differential operators

$$\partial^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i} \quad \text{and} \quad \partial^S = \partial^{\mathbb{1}_S} = \prod_{i \in S} \partial_i.$$

Note that if f is homogeneous of degree d, then $\partial^\alpha f$ is homogenous of degree $d - |\alpha|$ where $|\alpha| = \sum_{i=1}^n \alpha_i$.

A symmetric matrix $Q \in \mathbb{R}^{n \times n}$, alternatively viewed as a quadratic form $z \mapsto z^\intercal Q z$, is positive semidefinite if $v^\intercal Q v \geq 0$ for all $v \in \mathbb{R}^n$ and negative semidefinite if $v^\intercal Q v \leq 0$ for all $v \in \mathbb{R}^n$. If these inequalities are strict for $v \neq 0$, then Q is positive or negative definite, respectively. There are several equivalent definitions. In particular, a matrix is positive semidefinite if and only if all of its eigenvalues are nonnegative, which occurs if and only if the all its principal minors are nonnegative. Since Q is negative semidefinite if and only if $-Q$ is positive semidefinite, these translate into analogous characterizations of negative semidefinite-ness.

2.1 Matroids

Formally, a **matroid** $M = ([n], \mathcal{I})$ consists of a ground set $[n]$ and a nonempty collection $\mathcal{I}$ of *independent* subsets of $[n]$ satisfying the following two conditions:

(1) If $S \subseteq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$.

(2) If $S, T \in \mathcal{I}$ and $|T| > |S|$, then there exists an element $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{I}$.

The **rank**, denoted by $\text{rank}(S)$, of a subset $S \subseteq [n]$ is the size of the largest independent set contained in S and the rank of M is defined as $\text{rank}([n])$. An element $i \in [n]$ is called a **loop** if $\{i\} \notin \mathcal{I}$, and two elements $i, j \in [n]$ are called parallel if neither is a loop and $\text{rank}(\{i, j\}) = 1$. One can check that parallelism defines an equivalence relation on the non-loops of M, which partitions the set of non-loops into parallelism classes.

For a matroid M and an independent set $S \in \mathcal{I}$, the **contraction**, M/S, of M by S is the matroid on ground set $[n] \setminus S$ with independent sets $\{T \subseteq [n] \setminus S \mid S \cup T \in \mathcal{I}\}$. In particular, the rank of M/S equals $\text{rank}(M) - |S|$. See [Oxl11] for more details and general reference.

2.2 Log-concave polynomials

In [AOV18], it was shown that a homogeneous polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ with nonnegative coefficients is log-concave at a point $z = a$ if and only if its Hessian $\nabla^2 f$ has at most one positive eigenvalue at $z = a$. One can relate this directly to the negative semidefinite-ness of the Hessian of $\log(f)$. Indeed, there are several useful equivalent characterizations of this condition:

Lemma 2.1. *Let $f \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree $d \geq 2$ with nonnegative coefficients. Fix a point $a \in \mathbb{R}^n_{\geq 0}$ with $f(a) \neq 0$, and let $Q = \nabla^2 f\big|_{z=a}$. The following are equivalent:*

(1) *f is log-concave at $z = a$,*

(2) *$z \mapsto z^\intercal Q z$ is negative semidefinite on $(Qa)^\perp$,*

(3) *$z \mapsto z^\intercal Q z$ is negative semidefinite on $(Qb)^\perp$ for every $b \in \mathbb{R}^n_{\geq 0}$ such that $Qb \neq 0$,*

(4) *$z \mapsto z^\intercal Q z$ is negative semidefinite on some linear space of dimension $n - 1$, and*

(5) *the matrix $(a^\intercal Q a)Q - (Qa)(Qa)^\intercal$ is negative semidefinite.*

For $d \geq 3$, these are also equivalent to the condition

(6) *$D_a f$ is log-concave at $z = a$.*

One can check that this condition is also equivalent to Q having at most one positive eigenvalue, but we do not rely on this fact and leave its proof to the interested reader.

Proof. Euler's identity states that for a homogeneous polynomial g of degree d, $\sum_{i=1}^{n} z_i \partial_i g$ equals $d \cdot g$. Using this on f and $\partial_j f$ gives that $Qa = (d-1) \cdot \nabla f(a)$ and $a^\intercal Q a = d(d-1) \cdot f(a)$. The Hessian of $\log(f)$ at $z = a$ then equals

$$\nabla^2(\log(f))\big|_{z=a} \;=\; \left(\frac{f \cdot \nabla^2 f - \nabla f \nabla f^\intercal}{f^2}\right)\Bigg|_{z=a} \;=\; d(d-1)\frac{a^\intercal Q a \cdot Q - \frac{d}{d-1}(Qa)(Qa)^\intercal}{(a^\intercal Q a)^2}.$$

We can also conclude that $a^\intercal Q a = d(d-1) \cdot f(a) > 0$ and that the vector Qa is nonzero.

($1 \Rightarrow 2$) If f is log-concave at $z = a$, then the Hessian of $\log(f(z))$ at $z = a$ is negative semidefinite. Restricted to the linear space $(Qa)^\perp = \{z \in \mathbb{R}^n \mid z^\intercal Q a = 0\}$, the formula above simplifies to $\frac{d(d-1)}{a^\intercal Q a} \cdot Q$, meaning that $z \mapsto z^\intercal Q z$ is negative semidefinite on this linear space.

($2 \Rightarrow 4$) Since Qa is nonzero, $(Qa)^\perp$ has dimension $n-1$.

($4 \Rightarrow 5$) Suppose that $z \mapsto z^\intercal Q z$ is negative semidefinite on an $(n-1)$-dimensional linear space L. Let $b \in \mathbb{R}^n$ and consider the $n \times 2$ matrix P with columns a and b. Then

$$P^\intercal Q P = \begin{bmatrix} a^\intercal Q a & a^\intercal Q b \\ b^\intercal Q a & b^\intercal Q b \end{bmatrix}.$$

If P has rank one, then so does $P^\intercal Q P$, meaning that $\det(P^\intercal Q P) = 0$. Otherwise P has rank two and its column-span intersects L nontrivially. This means there is a vector $v \in \mathbb{R}^2$ for which $Pv \in L$ is nonzero and $(Pv)^\intercal Q (Pv) \leq 0$. From this we see that $P^\intercal Q P$ is not positive definite. On the other hand, since the diagonal entry $a^\intercal Q a$ is positive, $P^\intercal Q P$ is not negative definite. In either case, we then find that

$$\det(P^\intercal Q P) = (a^\intercal Q a) \cdot (b^\intercal Q b) - (b^\intercal Q a) \cdot (a^\intercal Q b) \leq 0.$$

Thus $b^\intercal((a^\intercal Q a) \cdot Q - (Qa)(Qa)^\intercal)b \leq 0$ for all $b \in \mathbb{R}^n$.

($5 \Rightarrow 1$) Suppose $(a^\intercal Q a) \cdot Q - (Qa)(Qa)^\intercal$ is negative semidefinite. Further subtracting $\frac{1}{d-1}(Qa)(Qa)^\intercal$ and scaling by the positive number $\frac{d(d-1)}{(a^\intercal Q a)^2}$ results in $\nabla^2(\log(f))\big|_{z=a}$, as above, which must therefore also be negative semidefinite.

($3 \Leftrightarrow 4$) Both conditions depend only on the matrix Q. We can then use the equivalence ($2 \Leftrightarrow 3$) for the point $z = b$ and the quadratic polynomial $f(z) = \frac{1}{2} z^\intercal Q z$, whose Hessian at any point is the matrix Q.

($1 \Leftrightarrow 6$) For $d \geq 3$, $D_a f$ is homogeneous of degree ≥ 2. Euler's identity applied to $\partial_i \partial_j f$ shows that the Hessian of $D_a f$ at $z = a$ is a scalar multiple of the Hessian of f at $z = a$, namely $(d-2)\, \nabla^2 f\big|_{z=a}$. Thus by the equivalence ($1 \Leftrightarrow 4$), $D_a f$ is log-concave at a if and only if f is. $\square$

2.3 Completely log-concave polynomials

One of the basic operations that preserves complete log-concavity is an affine change of coordinates. This was first proved in [AOV18], but for completeness we include the proof here.

Lemma 2.2. *If $f \in \mathbb{R}[z_1, \ldots, z_n]$ is completely log-concave and $T : \mathbb{R}^m \to \mathbb{R}^n$ is an affine transform such that $T(\mathbb{R}^m_{\geq 0}) \subseteq \mathbb{R}^n_{\geq 0}$, then $f(T(y_1, \ldots, y_m)) \in \mathbb{R}[y_1, \ldots, y_m]$ is completely log-concave.*

Proof. First, we prove that if f is a log-concave polynomial, then $f \circ T = f(T(y_1, \ldots, y_m))$ is also log-concave. By assumption for any $u, v \in \mathbb{R}^m_{\geq 0}$, we have $T(u), T(v) \in \mathbb{R}^n_{\geq 0}$. Thus for any $0 \leq \lambda \leq 1$,

$$f(T(\lambda u + (1-\lambda)v)) \;=\; f(\lambda T(u) + (1-\lambda)T(v)) \;\geq\; f(T(u))^{\lambda} f(T(v))^{1-\lambda}.$$

Therefore $f \circ T$ is log-concave.

Now suppose that f is completely log-concave and let $v_1, \ldots, v_k \in \mathbb{R}^m_{\geq 0}$. Since $T(\mathbb{R}^m_{\geq 0}) \subseteq \mathbb{R}^n_{\geq 0}$ and T is affine, $T(x) = Ax + b$ for some $A \in \mathbb{R}^{n \times m}_{\geq 0}$ and $b \in \mathbb{R}^n_{\geq 0}$. In particular, $Av_1, \ldots, Av_k \in \mathbb{R}^n_{\geq 0}$, which means that $D_{Av_1} \ldots D_{Av_k} f$ is log-concave over $\mathbb{R}^n_{\geq 0}$. By the chain rule for differentiation, we have

$$D_{v_1} \ldots D_{v_k}(f \circ T) = (D_{Av_1} \ldots D_{Av_k} f) \circ T.$$

Since composition with T preserves log-concavity, this polynomial is log-concave over $\mathbb{R}^m_{\geq 0}$. □

3 Reduction to quadratics

As the main result of this section we will show that, under some mild restrictions, to check whether a homogeneous polynomial is completely log-concave, it suffices to check the conditions in Definition 1.3 for $k = d - 2$ and $v_1, \ldots, v_k \in \{\mathbb{1}_{\{1\}}, \ldots, \mathbb{1}_{\{n\}}\}$. Then $D_{v_1} \cdots D_{v_k} f$ has the form $\partial^{\alpha} f$ where α_j is the number of vectors v_k equal to $\mathbb{1}_{\{j\}}$. This provides a powerful tool to check complete log-concavity. The mild restriction we impose is *indecomposability* of f and its derivatives.

Definition 3.1. A polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is **indecomposable** if it cannot be written as $f_1 + f_2$, where f_1, f_2 are nonzero polynomials in disjoint sets of variables. Equivalently, if we form a graph with vertices $\{i \mid \partial_i f \neq 0\}$ and edges $\{(i,j) \mid \partial_i \partial_j f \neq 0\}$, then f is indecomposable if and only if this graph is connected.

Now we are ready to state the main result of this section.

Theorem 3.2. *Let f be a homogeneous polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ of degree $d \geq 2$ with nonnegative coefficients. If the following two conditions hold, then f is completely log-concave:*

i) For all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq d-2$, the polynomial $\partial^{\alpha} f$ is indecomposable.

ii) For all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = d-2$, the quadratic polynomial $\partial^{\alpha} f$ is log-concave over $\mathbb{R}^n_{\geq 0}$.

The converse of the above statement is also true, namely, every completely polynomial is indecomposable, but we defer the proof of this fact to a future article.

We build up to the proof of this theorem with a series of lemmas. The first is a criterion for the sum of two log-concave polynomials to be log-concave. We will then use this to prove that if a polynomial f is indecomposable and all of its partial derivatives $\partial_i f$ are log-concave, then it itself must be log-concave. The proof of Theorem 3.2 then follows by an induction on the degree.

Lemma 3.3. *Let $f, g \in \mathbb{R}[z_1, \ldots, z_n]$ be homogenous with nonnegative coefficients satisfying $D_b f = D_c g \neq 0$ for some vectors $b, c \in \mathbb{R}^n_{\geq 0}$. If f and g are log-concave on $\mathbb{R}^n_{\geq 0}$ then so is $f + g$.*

Proof. The assumption that $D_b f = D_c g \neq 0$ means that f and g have the same degree d. We proceed by induction on d. If $d = 1$, then $f + g$ is a linear form with nonnegative coefficients, which is automatically log-concave on $\mathbb{R}^n_{\geq 0}$. Now suppose $d \geq 2$. Fix $a \in \mathbb{R}^n_{>0}$ and let $Q_1 = \nabla^2 f(a)$ and $Q_2 = \nabla^2 g(a)$. Then $D_b f = D_c g$ implies that for each $i = 1, \dots, n$,

$$(Q_1 b)_i \;=\; (\partial_i D_b f)\big|_{z=a} \;=\; (\partial_i D_c g)\big|_{z=a} \;=\; (Q_2 c)_i,$$

showing that $Q_1 b = Q_2 c$. Since $D_b f$ has nonnegative coefficients and is not identically zero, we also have that $D_b f(a) \neq 0$, meaning that $Q_1 b \neq 0$. By Lemma 2.1 (1 $\Rightarrow$ 3) and the log-concavity of f and g, each quadratic form $z \mapsto z^\intercal Q_i z^\intercal$ is negative semidefinite on $(Q_1 b)^\perp = (Q_2 c)^\perp$. It follows that their sum $z \mapsto z^\intercal (Q_1 + Q_2) z$ given by the matrix $Q_1 + Q_2 = \nabla^2 (f+g)\big|_{z=a}$ is also negative semidefinite on this $(n-1)$-dimensional linear space, so by Lemma 2.1 (4 $\Rightarrow$ 1), $f + g$ is log-concave at $z = a$. $\square$

Lemma 3.4. *Let $f \in \mathbb{R}[z_1, \dots, z_n]$ be homogeneous of degree $d \geq 3$ and indecomposable with nonnegative coefficients. If $\partial_i f$ is log-concave on $\mathbb{R}^n_{\geq 0}$ for every $i = 1, \dots, n$, then so is $D_a f$ for every $a \in \mathbb{R}^n_{\geq 0}$.*

Proof. If $\partial_i f$ is identically zero for some i, then we can consider f as a polynomial in the other variables. Without loss of generality, we can assume that $\partial_i f$ is nonzero for all i, and if necessary relabel $z_1, \dots, z_n$ so that that for every $2 \leq j \leq n$, there exists $i < j$ for which $\partial_i \partial_j f$ is non-zero. The latter follows from indecomposability.

Fix $a \in \mathbb{R}^n_{>0}$. We will show that $D_a f$ is log-concave on $\mathbb{R}^n_{\geq 0}$. We show by induction on k that for any $1 \leq k \leq n$, $\sum_{i=1}^k a_i \partial_i f$ is log-concave on $\mathbb{R}^n_{\geq 0}$. The case $k = 1$ follows by assumption. For $1 \leq k < n$, let b denote the truncation of a to its first k coordinates, $b = (a_1, \dots, a_k, 0, \dots, 0)$ and let c denote the vector $a_{k+1} \mathbb{1}_{\{k+1\}}$. By induction both $D_b f$ and $D_c f$ are log-concave, and

$$D_c D_b f \;=\; D_b D_c f \;=\; \sum_{i=1}^{k} a_i a_{k+1} \partial_i \partial_{k+1} f.$$

Since the coefficients of each summand are nonnegative and $\partial_i \partial_{k+1} f$ is non-zero for some $1 \leq i \leq k$, this sum is also non-zero. Then by Lemma 3.3, $D_b f + D_c f = \sum_{i=1}^{k+1} a_i \partial_i f$ is log-concave on $\mathbb{R}^n_{\geq 0}$. For $k = n-1$, this is exactly $D_a f$. Taking closures then shows that $D_a f$ is log-concave on $\mathbb{R}^n_{\geq 0}$ for all $a \in \mathbb{R}^n_{\geq 0}$. $\square$

Proof of Theorem 3.2. We induct on $d = \deg(f)$. The case $d = 2$ is clear, so let $d \geq 3$. For any positive vector $v \in \mathbb{R}^n_{>0}$, $D_v f$ is also indecomposable. Indeed for any homogeneous polynomial g of degree ≥ 1 with nonnegative coefficients (such as $\partial_i f$ and $\partial_i \partial_j f$), $D_v g$ is identically zero if and only if g is.

By taking closure, it suffices to show that for vectors $v_1, \dots, v_k \in \mathbb{R}^n_{>0}$, the polynomial $D_{v_1} \cdots D_{v_k} f$ is log-concave on $\mathbb{R}^n_{\geq 0}$. If $k \geq d-1$, then $D_{v_1} \cdots D_{v_k} f$ is either identically zero or linear with nonnegative coefficients, in which case it is log-concave on $\mathbb{R}^n_{\geq 0}$, so we take $0 \leq k \leq d-2$. If $k = 0$, then to show that f is log-concave at a point $a \in \mathbb{R}^n_{\geq 0}$, by Lemma 2.1 (6 $\Rightarrow$ 1), it suffices to show that $D_a f$ is log-concave at $z = a$. This reduces the case $k = 0$ to the case $k = 1$.

Suppose $1 \leq k \leq d-2$. By induction $\partial_j f$ is completely log-concave for all $j = 1, \dots, n$, and hence $D_{v_1} \cdots D_{v_{k-1}} \partial_j f = \partial_j D_{v_1} \cdots D_{v_{k-1}} f$ is log-concave on $\mathbb{R}^n_{\geq 0}$. Since $D_{v_1} \cdots D_{v_{k-1}} f$ is indecomposable and has degree $d - k + 1 \geq 3$, it follows from Lemma 3.4 that $D_{v_1} \cdots D_{v_k} f$ is log-concave on $\mathbb{R}^n_{\geq 0}$. $\square$

4 Complete log-concavity of independence polynomials

In this section, we use Theorem 3.2 to prove that the homogenization of the generating polynomial of the independent sets of a matroid is completely log-concave. In the following section we use a restriction of this to derive Mason's conjecture.

Theorem 4.1. *For any matroid $M = ([n], \mathcal{I})$, the polynomial*

$$g_M(y, z_1, \dots, z_n) \;=\; \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} z_i$$

in $\mathbb{R}[y, z_1, \dots, z_n]$ is completely log-concave.

We prove this by looking at quadratic derivatives of g_M.

Lemma 4.2. *For any matroid $M = ([n], \mathcal{I})$, the quadratic polynomial $\partial_y^{n-2} g_M$ is log-concave on $\mathbb{R}^{n+1}_{\geq 0}$.*

Proof. After taking derivatives and rescaling, we see that

$$q \;=\; \frac{\partial_y^{n-2} g_M}{(n-2)!} \;=\; \frac{n(n-1)}{2} \cdot y^2 + (n-1) \cdot \sum_{\{i\} \in \mathcal{I}} y z_i + \sum_{\{i,j\} \in \mathcal{I}} z_i z_j.$$

Let Q denote the Hessian $\nabla^2 q$ of q. Note that columns and rows of $\nabla^2 q$ corresponding to loops in M are zero, and the log-concavity of q only depends on the principal submatrix of Q indexed by non-loops. In this spirit and in a slight abuse of notation, we use $\mathbb{1}$ within this proof to denote the indicator vector of the non-loops of M. Then we find that

$$\nabla^2 q \;=\; Q \;=\; \begin{bmatrix} n(n-1) & (n-1)\mathbb{1}^\intercal \\ (n-1)\mathbb{1} & B \end{bmatrix},$$

where B is an $n \times n$ matrix with $B_{ij} = 1$ when $\{i, j\}$ has rank two in M and $B_{ij} = 0$ otherwise. Since q is quadratic, its Hessian does not depend on any evaluation, so q is log-concave on $\mathbb{R}^{n+1}_{\geq 0}$ if and only if it is log-concave at the point $a = (1, 0, \dots, 0)$. By Lemma 2.1 (1 ⇔ 5), this happens if and only if the matrix

$$(a^\intercal Q a) Q - (Qa)(Qa)^\intercal \;=\; n(n-1)Q - (n-1)^2 \begin{bmatrix} n \\ \mathbb{1} \end{bmatrix} \begin{bmatrix} n \\ \mathbb{1} \end{bmatrix}^\intercal \;=\; (n-1) \begin{bmatrix} 0 & 0 \\ 0 & nB - (n-1)\mathbb{1}\mathbb{1}^\intercal \end{bmatrix}$$

is negative semidefinite. Thus it suffices to show that $nB - (n-1)\mathbb{1}\mathbb{1}^\intercal$ is negative semidefinite. As M is a matroid, the matroid partition property tells us that the nonloops of M may be partitioned into equivalence classes of parallel elements $P_1, \dots, P_c$. This lets us rewrite the matrix B as

$$B \;=\; \mathbb{1}\mathbb{1}^\intercal - \sum_{i=1}^{c} \mathbb{1}_{P_i} \mathbb{1}_{P_i}^\intercal \quad \text{and} \quad nB - (n-1)\mathbb{1}\mathbb{1}^\intercal \;=\; \mathbb{1}\mathbb{1}^\intercal - n \cdot \sum_{i=1}^{c} \mathbb{1}_{P_i} \mathbb{1}_{P_i}^\intercal.$$

We can now check that this matrix is negative semidefinite. Let $x \in \mathbb{R}^n$ and consider

$$x^\intercal (nB - (n-1)\mathbb{1}\mathbb{1}^\intercal) x \;=\; (\mathbb{1}^\intercal x)^2 - n \cdot \sum_{i=1}^{c} \left(\mathbb{1}_{P_i}^\intercal x \right)^2.$$

Since $P_1, \dots P_c$ partition the non-loops of M, $\mathbb{1}$ equals $\sum_{i=1}^{c} \mathbb{1}_{P_i}$. For any real numbers $u_1, \dots, u_c$, the Cauchy-Schwarz inequality implies that $(\sum_{i=1}^{c} u_i)^2 \leq c \cdot \sum_{i=1}^{c} u_i^2$. This then gives that

$$(\mathbb{1}^\intercal x)^2 \;=\; \left(\sum_{i=1}^{c} \mathbb{1}_{P_i}^\intercal x\right)^2 \;\leq\; c \cdot \sum_{i=1}^{c} \left(\mathbb{1}_{P_i}^\intercal x\right)^2 \;\leq\; n \cdot \sum_{i=1}^{c} \left(\mathbb{1}_{P_i}^\intercal x\right)^2.$$

For the last inequality, we use the fact the number of equivalence classes c of nonloops of M is at most n. It follows that $x^\intercal(nB - (n-1)\mathbb{1}\mathbb{1}^\intercal)x \leq 0$ for all x and by Lemma 2.1, q is log-concave on $\mathbb{R}_{\geq 0}^{n+1}$. □

Proof of Theorem 4.1. We will use the criterion in Theorem 3.2 to show complete log-concavity.

Here we use ∂_i to mean ∂_{z_i} and for $\alpha \in \mathbb{Z}_{\geq 0}^n$, ∂^α to denote $\prod_{i=1}^n \partial_i^{\alpha_i}$. We need to show that for every $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathbb{Z}_{\geq 0}^n$ with $k + |\alpha| \leq n-2$, the polynomial $\partial_y^k \partial^\alpha g_M$ is indecomposable and that for $k + |\alpha| = n-2$ it is log-concave.

Note that if $\alpha_i \geq 2$ for any i, then $\partial^\alpha g_M$ is zero, so we may consider $\alpha = \mathbb{1}_J$ for some $J \subseteq [n]$. Similarly, if J is not an independent set of M, then $\partial^\alpha g_M = \partial^J g_M = 0$. Therefore is suffices to consider $\alpha = \mathbb{1}_J$ for $J \in \mathcal{I}$. In this case, the derivative $\partial^J g_M$ equals the polynomial $g_{M/J}$ of the contraction M/J, namely

$$\partial^J g_M \;=\; \sum_{I \in \mathcal{I} : J \subseteq I} y^{n-|I|} \prod_{i \in I \setminus J} z_i \;=\; \sum_{I \in \mathcal{I} : J \subseteq I} y^{n-|J|-|I\setminus J|} \prod_{i \in I \setminus J} z_i \;=\; g_{M/J}.$$

Recall that M/J is a matroid on ground set $[n] \setminus J$ with independent sets $\{I \setminus J \mid J \subseteq I \in \mathcal{I}\}$.

First we check indecomposability of $\partial_y^k \partial^J g_M = \partial_y^k g_{M/J}$. Note that if $i \in [n] \setminus J$ is a loop of M/J, then the variable z_i does not appear in $g_{M/J}$ and $\partial_i g_{M/J} = 0$. Similarly, $\partial_i g_{M/J}$ is zero for all $i \in J$. Otherwise, the monomial $y^{n-|J|-1-k} z_i$ appears in $\partial_y^k g_{M/J}$ with non-zero coefficient. Since $k + |J| \leq n-2$, it follows that $\partial_y \partial_i g_{M/J}$ is non-zero. In particular, the graph formed in Definition 3.1 is a star centered at the variable y, and thus connected. Therefore $\partial_y^k \partial^J g_M$ is indecomposable.

Now suppose $k + |J| = n-2$. Since M/J is a matroid on $n - |J|$ elements, Lemma 4.2 imples that $\partial_y^{n-|J|-2} g_{M/J} = \partial_y^k \partial^J g_M$ is log-concave on $\mathbb{R}_{\geq 0}^{n+1}$. All together with Theorem 3.2, this implies that the polynomial g_M is completely log-concave. □

Corollary 4.3. *Given a matroid $M = ([n], \mathcal{I})$ with $\mathcal{I}_k$ independent sets of size k, the bivariate polynomial*

$$f_M(y,z) \;=\; \sum_{k=0}^{r} \mathcal{I}_k \, y^{n-k} z^k,$$

is completely log-concave.

Proof. Note that f_M is the restriction of the completely log-concave polynomial g_M to $z_i = z$ for all $i \in [n]$. Since the image of $\mathbb{R}_{\geq 0}^2$ under the linear map $(y,z) \mapsto (y, z, \dots, z)$ is contained in $\mathbb{R}_{\geq 0}^{n+1}$, Lemma 2.2 implies that $f_M(y,z) = g_M(y,z,\dots,z)$ is completely log-concave. □

5 Proof of Mason's conjecture

We use the following proposition, which was first observed by Gurvits [Gur09], and give a short proof for the sake of completeness.

Proposition 5.1 (Proposition 2.7 from [Gur09]). *If* $f = \sum_{k=0}^{n} c_k y^{n-k} z^k \in \mathbb{R}[y,z]$ *is completely log-concave, then the sequence* $c_0, \ldots, c_n$ *is ultra log-concave. That is, for every* $1 < k < n$*,*

$$\left(\frac{c_k}{\binom{n}{k}}\right)^2 \geq \frac{c_{k-1}}{\binom{n}{k-1}} \cdot \frac{c_{k+1}}{\binom{n}{k+1}}.$$

Remark 5.2. In [Gur09], Gurvits assumes *strong log-concavity* and also shows the converse. In a future article, we show the equivalence of strong and complete log-concavity for homogeneous polynomials.

Proof. Since f is completely log-concave, for any $1 < k < n$, the quadratic $q(y,z) = \partial_y^{n-k-1}\partial_z^{k-1} f$ is log-concave over $\mathbb{R}^2_{\geq 0}$. Notice that for any $0 \leq m \leq n$,

$$\partial_y^{n-m}\partial_z^{m} f \;=\; (n-m)!\; m!\; c_m \;=\; n!\; \frac{c_m}{\binom{n}{m}}.$$

Using this for $m = k-1, k, k+1$, we can write the Hessian of q as

$$\nabla^2 q \;=\; \begin{bmatrix} \partial_y^2 q & \partial_y\partial_z q \\ \partial_y\partial_z q & \partial_z^2 q \end{bmatrix} \;=\; n!\; \begin{bmatrix} c_{k-1}/\binom{n}{k-1} & c_k/\binom{n}{k} \\ c_k/\binom{n}{k} & c_{k+1}/\binom{n}{k+1} \end{bmatrix}.$$

Since q is log-concave on $\mathbb{R}^2_{\geq 0}$, by Lemma 2.1 its Hessian cannot be positive or negative definite. Its determinant is therefore non-positive. This gives the desired inequality:

$$0 \;\geq\; \det(\nabla^2 q) \;=\; (n!)^2 \left(\frac{c_{k-1}}{\binom{n}{k-1}} \cdot \frac{c_{k+1}}{\binom{n}{k+1}} - \left(\frac{c_k}{\binom{n}{k}}\right)^2 \right).$$

□

The strong version of Mason's conjecture, Theorem 1.2, then follows from Corollary 4.3.

References

[AHK18] Karim Adiprasito, June Huh, and Eric Katz. "Hodge theory for combinatorial geometries". In: *Annals of Mathematics* 188.2 (2018), pp. 381–452.

[AOV18] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. "Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids". In: *FOCS*. to appear. 2018.

[Ard18] Federico Ardila. "The geometry of matroids". In: *Notices Amer. Math. Soc.* 65.8 (2018), pp. 902–908. ISSN: 0002-9920.

[BBL09] Julius Borcea, Petter Brändén, and Thomas M. Liggett. "Negative dependence and the geometry of polynomials". In: *Journal of the American Mathematical Society* 22.2 (2009), pp. 521–567.

[Dow80] Thomas A Dowling. "On the independent set numbers of a finite matroid". In: *Annals of Discrete Mathematics*. Vol. 8. Elsevier, 1980, pp. 21–28.

[Gur09] Leonid Gurvits. "On multivariate Newton-like inequalities". In: *Advances in Combinatorial Mathematics*. Springer, 2009, pp. 61–78.

[HK12] June Huh and Eric Katz. "Log-concavity of characteristic polynomials and the Bergman fan of matroids". In: *Math. Ann.* 354 (2012), pp. 1103–1116.

[HS89] Yahya Ould Hamidoune and Isabelle Salaün. "On the independence numbers of a matroid". In: *Journal of Combinatorial Theory Series B* 47.2 (1989), pp. 146–152.

[HSW18] June Huh, Benjamin Schröter, and Botong Wang. "Correlation bounds for fields and matroids". https://arxiv.org/abs/1806.02675. 2018.

[KN10] Jeff Kahn and Michael Neiman. "Negative correlation and log-concavity". In: *Random Structures & Algorithms* 37.3 (2010), pp. 367–388.

[KN11] Jeff Kahn and Michael Neiman. "A strong log-concavity property for measures on Boolean algebras". In: *Journal of Combinatorial Theory Series A* 118.6 (2011), pp. 1749–1760.

[Len13] Matthias Lenz. "The f-vector of a representable-matroid complex is log-concave". In: *Advances in Applied Mathematics* 51.5 (2013), pp. 543–545.

[Mah85] Carolyn Mahoney. "On the unimodality of the independent set numbers of a class of matroids". In: *Journal of Combinatorial Theory Series B* 39.1 (1985), pp. 77–85.

[Mas72] John Mason. "Matroids: unimodal conjectures and Motzkin's theorem". In: *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst.)* Inst. Math. Appl., Southend-on-Sea, 1972, pp. 207–220.

[Oxl11] James Oxley. *Matroid theory*. Second. Vol. 21. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, pp. xiv+684. ISBN: 978-0-19-960339-8. DOI: 10.1093/acprof:oso/9780198566946.001.0001.

[Pem00] Robin Pemantle. "Towards a theory of negative dependence". In: *Journal of Mathematical Physics* 41 (3 2000), pp. 1371–1390.

[Sey75] Paul Seymour. "Matroids, hypergraphs, and the max-flow min-cut theorem". PhD thesis. University of Oxford, 1975.

[SW75] Paul Seymour and Dominic Welsh. "Combinatorial applications of an inequality from statistical mechanics". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 77.3 (1975), pp. 485–495.

[Wag08] David G. Wagner. "Negative Correlated Random Variables and Mason's Conjecture for Independent Sets in Matroids". In: *Annals of Combinatorics* 12.2 (2008), pp. 211–239.

[Zha85] Cui Kui Zhao. "A conjecture on matroids". In: *Neimenggu Daxue Xuebao* 16.3 (1985), pp. 321–326.