Equilibrium Pricing with Positive Externalities

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Abstract

We study the problem of selling an item to strategic buyers in the presence of positive *historical externalities*, where the value of a product increases as more people buy and use it. This increase in the value of the product is the result of resolving bugs or security holes after more usage. We consider a continuum of buyers that are partitioned into *types* where each type has a valuation function based on the actions of other buyers. Given a fixed sequence of prices, or *price trajectory*, buyers choose a day on which to purchase the product, i.e., they have to decide whether to purchase the product early in the game or later after more people already own it. We model this strategic setting as a game, study existence and uniqueness of the equilibria, and design an FPTAS to compute an approximately revenue-maximizing pricing trajectory for the seller in two special cases: the *symmetric* settings in which there is just a single buyer type, and the *linear* settings that are characterized by an initial type-independent bias and a linear type-dependent influenceability coefficient.

1 Introduction

Many products like software, electronics, or automobiles evolve over time. When a consumer considers buying such a product, he faces a tradeoff between buying a possibly sub-par early version versus waiting for a fully functional later version. Consider, for example, the dilemma facing a consumer who wishes to purchase the latest Windows operating system. By buying early, the consumer takes full advantage of all the new features. However, operating systems may have more bugs and security holes at the beginning, and hence a consumer may prefer to wait with the rational that, if more people already own the operating system, then more bugs will have already been uncovered and corrected. The key observation is, the more people that use the operating system, or any product for that matter, the more inherent value it accrues. In other words, the product exhibits a particular type of externality, a so-called *historical externality*[1].

How should a company price a product in the presence of historical externalities? A low introductory price may attract early adopters and hence help the company extract greater revenue from future customers. On the other hand, too low a price will result in significant revenue loss from the initial sales. Often, when faced with such a dilemma, a company will offer an initial promotional price at the product's release in a limited-time offer, and then raise the price after some time. For example, when releasing Windows 7,

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[1]Note that this is different from the more well-studied notion of externalities in the computer science literature where a product (e.g., a cell phone) accrues value as more consumers buy it simply because the product is used in conjunction with other consumers.

Microsoft announced a two-week pre-order option for the Home Premium Upgrade version at a discounted price of $50; thereafter the price rose to $120, where it has remained since the pre-sale ended on July 11th, 2009. Additionally, beta testers, who can be interpreted as consumers who "bought" the product even prior to release, received the release version of Windows 7 for free (as is often the case with software beta-testers).[2]

We study this phenomenon in the following stylized model: a monopolistic seller wishes to derive a pricing and marketing plan for a product with historical externalities. To this end, she commits to a price trajectory.[3] Potential consumers observe the price trajectory and make simultaneous decisions regarding the day on which they will buy the product (and whether to buy at all). The payoff of a consumer is a function of the day on which he bought the product, the price on that day, and the set of consumers who bought before him.

We focus on the non-atomic setting in which we have a continuum of consumers so that each consumer is infinitesimally small and therefore his own action has a negligible effect on the actions of others. Consumers are drawn from a (possibly infinite) set of types. These types capture varying behavior among consumer groups. For example, beta-testers may be fairly insensitive to bugs in an operating system as they are tech-savvy enough to guard against the resulting insecurities. Hence their value for the operating system is fairly insensitive to the set of previous consumers. On the other hand, less tech-savvy consumers, like the home consumer, may be very sensitive to bugs and hence have a value that is highly dependent on the set of previous consumers. The home consumer may also place more faith in previous consumers that are beta-testers than previous consumers that are home consumers as the beta-testers are more likely to uncover bugs, and hence the value function of a consumer may react differently to different types of previous consumers. To make our problem tractable and realistic, we assume that the value function depends only on the fractions of consumers of each type, and is increasing in these parameters. Thus, the home consumer's value for the operating system should not decrease as the number of beta-testers that buy it increases.

In this paper, we model this as a two-stage game in which the seller first commits to a price trajectory and then the consumers simultaneously choose when and whether to buy in the induced normal-form game among them. We study subgame perfect equilibria. First we explore the existence and uniqueness of the full information equilibria in the induced normal-form subgame among consumers. An action profile of this game specifies what fraction of each consumer type buys the product in any given day. The profile is an equilibrium if the payoff for a given type is equal and is non-negative on every day in which consumers of this type choose to purchase the product. Using Kakutani's fixed-point theorem [9], we show that for any price trajectory, equilibria exist so long as there are finitely many types and the value functions are continuous. We further show via an example that if the value functions are not continuous, then equilibria may not exist.

We then turn to the question of uniqueness. We focus on well-behaved equilibria in which consumers with non-negative utility always purchase the product (thus indifferent consumers purchase the product; these are precisely the equilibria we care about when computing revenue-maximizing price trajectories). In general multiple such equilibria may exist. However, in an aggregate model in which the value function of each consumer type depends only on the *aggregate* behavior of the population (i.e., the total fraction of

[2]Historical prices, announced upon the press release, can be found in archived versions of various technology news websites such as Ars Technia [15] and the Microsoft blog [12]. The current prices were accessed on Microsoft's website at the time of submission.

[3]Such commitments are observed in many settings especially at the outset of a new product (see the Windows 7 example described above) and have been assumed in prior models in the economics literature (see Section 1.1).

potential consumers that have bought the product and not the total fraction of various types), then we are able to show that when they exist the well-behaved equilibria of this game are unique in the sense that the fraction of purchases per-type-per-day is fixed among all equilibria. Thus while a given price trajectory may have many equilibria, the revenue-maximizing one is unique from a revenue perspective; i.e., when they exist the well-behaved equilibria both maximize revenue and are revenue-equivalent.

Finally, we search for the revenue-maximizing price trajectory. We address this question in settings in which we either have just one type or there are multiple types whose valuation functions are linear in the aggregate. These settings are special cases of the aggregate model discussed above and hence well-behaved equilibria exist and are unique.[4] For each price trajectory, we define its revenue to be the amount of money consumers spend on the product. We then design an FPTAS to find the revenue-maximizing price trajectory for a monopolistic seller in these settings. We do this via a reduction to a novel rectangle covering problem in which we must find the discounted area-maximizing set of rectangles that fit underneath a given curve.

In summary, we get the following main result for the settings described above: first, every price trajectory has well-behaved equilibria that are revenue-unique and revenue-maximal among all possible consumer equilibria. Second, it is possible to (approximately) compute the revenue-maximizing price trajectory. Hence, the strategy tuple in which the seller announces this (approximately) revenue-maximizing price trajectory and the consumers respond by playing a well-behaved equilibrium is an (approximately) subgame perfect equilibrium of our two-stage game.[5]

As an interesting consequence of our result, we find that the revenue maximizing price trajectory is an increasing and convex function, matching the intuition that the seller should attract a few early adopters with a low introductory price and then exploit the value they add by offering high prices to remaining consumers. We also note that the distribution of sales in the revenue maximizing equilibrium matches this intuition as well – it is also increasing and convex.

1.1 Related Work

Our work falls in the long line of literature investigating pricing and marketing of products that exhibit externalities [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16]. Among these, the paper of Bensaid and Lesne [4] is most closely related to our own work. Bensaid and Lesne [4] analyzed the pricing problem in the presence of historical externalities which they call *word of mouth externalities*, in which the customers benefit from past sales as it reduces the amount of uncertainty about the product's quality, and *learning by doing externalities*, in which the initial comments by consumers help the seller improve product causing its value to be a function of the quantity of past sales. The authors focus on linear forms for the externalities. They consider two cases depending on whether the seller is able to commit to a price trajectory. When the seller commits, they investigate the Nash equilibria of the induced game among consumers; when he does not they study subgame perfect equilibria. For both settings they consider either two price periods or an infinite sequence of price periods and observe that optimal price trajectories are increasing. The historical externalities that we study generalize the externalities of Bensaid and Lesne [4], and in this more general model, we solve for the optimal price sequence for any fixed number of price periods.

Most of the remaining externalities literature studies externalities in which consumers care about the

[4]Technically, our existence result only applies to finite types and does not establish the existence of a *well-behaved* equilibria; we show how to resolve these issues for the particular settings that we focus on here.

[5]We must also define what happens off the equilibrium path in subgame perfect equilibria; here we assume that for any price trajectory announced by the seller, the consumer respond by playing a well-behaved equilibrium.

total population of users of a product and hence their utility is affected by future sales as well as past sales. This model is more appropriate for products used in conjunction with other consumers – cell phones or social media, for example. Although the phenomenon studied is different from ours, some of the modeling assumptions in these papers are similar to ours. For example, in the economics literature, Cabral, Salant, and Woroch [5] also consider a seller that commits to a price trajectory and then observe that the revenue-maximizing price sequence with fully rational consumers (playing a Bayesian equilibrium) is increasing. Similar to our model, they study the pricing problem in the presence of a continuum of consumers.

In the computer science literature Akhlaghpour et al. [1] and Hartline et al. [8] study algorithmic questions regarding revenue maximization over social networks for products with externalities. However, their models assume naive behavior for consumers. Namely, they assume consumers act myopically, buying the product on the first day in which it offers them positive utility without reasoning about future prices and sales that could affect optimal buying behavior and long-term utility. Furthermore, Hartline et al. [8] allow the seller to use adaptive price discrimination. In contrast, we model consumers as fully rational agents that strategically choose the day on which to buy based on full information regarding all future states of the world and a sequence of public posted prices. While the correct model of pricing and consumer behavior probably lies somewhere between these two extremes, we believe studying fully rational consumers is an important first step in relaxing myopic assumptions.

2 Model

We wish to study the sale of a good by a monopolistic seller over k days to a set of potential consumers or buyers. We model our setting as a two-stage game whose players consist of the monopolistic seller and a continuum of potential consumers or buyers $b \in [0,1]$. The strategy of the seller is a *price trajectory* $p = (p_1, \dots, p_k)$ where $p_i \in \Re$ assigning a (possibly negative) price p_i to each day i.

The buyers are partitioned into n types $T_1, \dots, T_n$ where each T_t is a subinterval of $[0,1]$.[6] The strategy set $A = \{1, \dots, k\} \cup \{\emptyset\}$ indicates the day on which the product is bought ($\emptyset$ is used to indicate that the product was not purchased). Hence the strategy profile of the buyer population can be represented by a $(k+1) \times n$ matrix $X = \{X_{i,t}\}_{i=1,\dots,k+1; t=1,\dots,n}$ where entry $X_{i,t}$ indicates the fraction of buyers of type t that buy the product *before* day i, and we define $X_{1,t} = 0$ for all t. Note that by normalization $\sum_t X_{k+1,t} \leq 1$ and $1 - \sum_t X_{k+1,t}$ is the fraction of buyers that don't buy the product at any time. Corresponding to this matrix X we also define the *marginal strategy profile* matrix $x = \{x_{i,t}\}_{i=1,\dots,k; t=1,\dots,n}$ where $x_{i,t} = X_{i+1,t} - X_{i,t}$ is the fraction of type t buyers who buy on day i. In the special case when there is only 1 type, we use X_i as a scalar to denote the fraction of buyers who bought before day i and x_i as a scalar to denote the fraction of buyers who buy on day i.

Given a strategy profile X, we define the value of buyers of type t buying on day i by a value function $F_i^t(X_i)$ where X_i is the i'th row of X (hence buyers are indifferent to future buying decisions). Note the explicit dependence of F on time, which allows $F_i^t(X_i)$ to be different than $F_j^t(X_j)$, for $i \neq j$. The revenue-maximization results in Section **??** further assume that the dependence of $F_i^t(X_i)$ on i is of the form $F_i^t(X_i) = \beta^i F^t(X)$ for $\beta \in [0,1]$. This special case is of particular interest as the β factor models settings in which the value degrades over time due to, for example, a reduction in the novelty of the product.

Given a strategy profile X, the payoff of buyers of type t who buy on day i is defined to be $(F_i^t(X_i) - p_i)$. We additionally allow buyers to have a discount factor α such that their payoff is $(1-\alpha)^i (F_i^t(X_i) - p_i)$. Thus

[6] Later, we will generalize this to infinitely many types.

α represents the way in which agents discount future payoffs with respect to present payoffs. We say that a strategy profile X is a Nash equilibrium of the induced subgame given by price trajectory p, or equivalently $X \in NE(p)$, if for any buyer of type t who buys on day i we have $i \in \arg\max_j (F_j^t(X_j) - p_j)(1-\alpha)^j$, and the strategy is $\emptyset$ whenever the maximum is negative (in which case the buyer's payoff is zero). We call an equilibrium *well-behaved* if all indifferent buyers buy, i.e., a buyer does not buy if and only if his payoff $(1-\alpha)^i(F_i^t(X_i) - p_i)$ is negative on all days $1 \le i \le k$. We say that (p, X) is a (well-behaved) equilibrium if the profile X is a (well-behaved) Nash equilibrium for the subgame of price trajectory p. Equivalently, a marginal strategy profile x is a (well-behaved) Nash equilibrium for the subgame of price trajectory p if for any type t and day i we have $x_{i,t} > 0$ only if $i \in \arg\max_j (F_j^t(X_j) - p_j)(1-\alpha)^j$ and the value of this maximum is non-negative.

Given a price trajectory p and a marginal strategy profile x that arises in the subgame induced by p, we define the payoff of the seller to be the *revenue* of x for p, which is $R(p, x) = \sum_{i=1}^{k} \sum_{t=1}^{n} x_{i,t} p_i (1-\alpha)^i$. A *subgame perfect equilibrium* of the two-stage game is then a price trajectory p^* and a set of marginal strategy profiles x_p for each possible price trajectory p such that: (1) x_p is a Nash equilibrium of the subgame induced by p, and (2) p^* maximizes $R(p, x_p)$. The *outcome* of this subgame perfect equilibrium is (p^*, x_{p^*}) and its revenue is $R(p^*, x_{p^*})$.

We are interested in computing the outcome in a revenue-maximizing subgame perfect equilibrium. To do so, we must compute a price trajectory which maximizes the revenue of the seller in equilibrium. Note that this is equal to finding the best response of the seller given the strategies $\{x_p\}$ of the buyers. We solve this problem for special settings in which there exist revenue-maximizing well-behaved equilibria in $NE(p)$ for any price trajectory p, allowing us to maximize over them. These settings are as follows. For the purpose of these definitions, we will allow each buyer to have a unique type and hence there are infinitely many types. We will use $b \in [0, 1]$ to denote the type of buyer b.

Definition 1 *The Aggregate Model: The value function of each type in this model is a function of the aggregate behavior of the population and is invariant with respect to the behavior of each separate type. That is, the value function of buyer b is a function of X_i only, where X_i is a scalar indicating the total fraction of all buyers who buy before day i. In this instance, we overload the notation for the value function and let $F_i^b(X_i)$ indicate the value of buyer b (hence $F_i^b(\cdot)$ now maps the unit interval to the non-negative reals).*

Definition 2 *The Linear Model: This is a special case of the aggregate model which is defined by a function F_i, an initial bias I, and a function C so that the value of buyer b is $F_i^b(X_i) = I + C(b) \cdot F_i(X_i)$. We further define the commonly-known distribution $\mathcal{C} : \mathbb{R} \to [0, 1]$ such that $\mathcal{C}(c^*)$ indicates the fraction of buyers b with $C(b) \le c^*$.*

Definition 3 *The Symmetric Model: In this version we only have one type, that is, $F_i^b = F_i$ for all b.*

3 Characterizing Equilibria

3.1 Equilibrium Existence

In Appendix A.1, we give examples of games with a non-continuous valuation function in which no equilibria exist. Here, we prove that the equilibrium exists for continuous valuation functions F_i^t, $1 \le t \le$

$n, 1 \leq i \leq k$. To do so, we define a set-valued function on the space of marginal strategy profile matrices whose fixed point is an equilibrium of our game. To prove the existence of a fixed point, we use Kakutani's fixed point theorem (KFPT). Let $\phi : X \to 2^Y$ be a set-valued function, i.e, a function from X to the power set of Y. We say that ϕ has a closed graph if the set $\{(x, y) | y \in \phi(x)\}$ is a closed subset of $X \times Y$ in the product topology.

Theorem 1 *(Kakutani Fixed Point Theorem (KFPT) [9]) Let S be a non-empty, compact and convex subset of Euclidean space R^n for some n. Let $\phi : S \to 2^S$ be a set-valued function on S with a closed graph and the property that $\phi(x)$ is non-empty and convex for all $x \in S$. Then ϕ has a fixed point x such that $x \in \phi(x)$.*

We are now ready to prove the equilibrium existence. We define ϕ to be the correspondence that maps strategy profile matrices to the set of best-response matrices and then use KFPT to show that this mapping has a fixed point.

Theorem 2 *If valuation functions are increasing and continuous, then our game has an equilibrium.*
Proof : Let S be a subset of the Euclidean space $\mathbb{R}^{k \times n}$ consisting of all valid marginal profile matrices x. Also let $\mu(t)$ be the length of T_t for each type t (recall that T_t is a subinterval of $[0, 1]$). Each $x \in S$ is a marginal strategy profile matrix $x = (x_{1,1}, \cdots, x_{k,n})$, where $x_{i,t}$ is the fraction of type t buyers who choose to buy on day i, with the constraint that for each $1 \leq t \leq n$, the inequality $\sum_i x_{i,t} \leq \mu(t)$ holds. Define $\phi : S \to 2^S$ to be the function assigning each $x \in S$ the set of all marginal strategy profile matrices $y \in \phi(x)$ which are simultaneous best-responses to the profile x. Formally, $\phi(x)$ consists of all y satisfying the following conditions:

1. A buyer buys in y only if they get non-negative utility in x: $\sum_i y_{i,t} > 0$ only if there exists j such that $F_j^t(X_j) - p_j \geq 0$,
2. If some type has a positive utility in x, then they all buy in y: if $F_j^t(X_j) - p_j > 0$ then $\sum_i y_{i,t} \geq \mu(t)$,
3. If a buyer buys in y, then he does so on a day which gives him maximum utility in x: i, t, if $y_{i,t} > 0$ then $i \in \arg\max_j F_j^t(X_j) - p_j$.

If the conditions of the KFPT hold, we get a fixed point $x \in S$; i.e. a point x for which $x \in \phi(x)$. It is easy to check that any such fixed point is an equilibrium of our game. Now let us prove that the set S and the function ϕ satisfy the conditions of KFPT. Set S can be defined as the set of points $x \in \mathbb{R}^{k \times n}$ satisfying the following linear inequalities, i.e., $\forall i, t : x_{i,t} \geq 0$, and $\forall t : \sum_i x_{i,t} \leq \mu(t)$. As a result, S, being the intersection of half-spaces, is a polyhedron, and clearly is closed and convex. The set S is also bounded, because each $x_{i,t}$ lies in the interval $[0, \mu(t)]$. So S is a compact and convex subset of $\mathbb{R}^{k \times n}$.

Let x be an arbitrary point in S. The set $\phi(x)$ can be defined as the intersection of S, and a set of (possibly open) half-spaces defined by linear inequalities listed in the conditions above. Thus, $\phi(x)$ is a convex set. It is also nonempty as each buyer of each type t has a well-defined set of best-responses to X, the cumulative corresponding to marginal profile x, which is either some day j if $F_j^t(X_j) - p_j \geq 0$ or the empty strategy (not buying) otherwise.

It only remains to show that the graph of ϕ is a closed subset of $\mathbb{R}^{2(k \times n)}$. We will show that each (x, y) lying outside the graph is contained in an open neighborhood which also lies outside the graph. This neighborhood will be of the form $A \times B$, where A is an open neighborhood of x and B is an open neighborhood of y. Since (x, y) is not in the graph, either (x, y) is not in $S \times S$ or y does not satisfy one of the conditions defining $\phi(x)$. In the former case, since $S \times S$ is closed, we can find a suitable neighborhood of (x, y) having no intersection with $S \times S$ and by extension the graph of ϕ.

So assume that y does not satisfy at least one of the constraints defining $\phi(x)$. Let $U : \mathbb{R}^{k \times n} \to \mathbb{R}^{k \times n}$ be the function assigning each $x \in S$ the matrix of utilities $\{u_{i,t}\}$, where $u_{i,t} = F_i^t(X_i) - p_i$ denotes the utility of buying on day i for a buyer of type t. Assuming the valuation functions are continuous and increasing, U and hence U^{-1} is continuous and invertible. As $y \notin \phi(x)$, there is some type t such that either:

1. $\sum_j y_{j,t} > 0$ and for all days j, $u_{j,t} < 0$: Let $A = \{U^{-1}(\{u_{i,t}\}) \mid \forall j : u_{j,t} < 0\}, B = \{\{y_{i,t}\} \mid \sum_j y_{j,t} > 0\}$,
2. Or $\max_j u_{j,t} > 0$ and $\sum_i y_{i,t} < \mu(t)$: Let $A = \{U^{-1}(\{u_{i,t}\}) \mid \exists i : u_{i,t} > 0\}, B = \{\{y_{i,t}\} \mid \sum_i y_{i,t} < \mu(t)\}$,
3. Or there exists $j^* \notin \arg\max_j u_{j,t}$ and $y_{j^*,t} > 0$: Let $A = \{U^{-1}(\{u_{i,t}\}) \mid \exists i : u_{i,t} > u_{j^*,t}\}, B = \{\{y_{i,t}\} \mid y_{j^*,t} > 0\}$.

Then $A \times B$ is an open neighborhood containing (x, y) since U^{-1} is continuous and invertible and we are applying it to an open subset of the domain in each case. Also, $A \times B$ has no intersection with the graph of ϕ, which proves that the graph of ϕ is closed. Hence all assumptions of the KFPT hold and we have an equilibrium. □

We note that the above proof holds only for a finite number of types. However, for the special setting of linear valuation functions, Appendix E shows that the existence statement still holds. Also note that while the above theorem proves existence of equilibria, it does not guarantee that the equilibria are necessarily well-behaved (i.e., all indifferent buyers buy at some point). Well-behaved equilibria may not exist. Furthermore, even when they exist they do not necessarily maximize revenue. However, for the linear and symmetric models that we focus on, we can show that well-behaved equilibria do in fact exist and maximize revenue. The existence results are presented in Appendix F; the results in Section 4 imply the revenue-maximizing property.

3.2 Uniqueness

Example 21 in Appendix A.2 shows that the game might have more than one equilibria, with different revenues for the seller, if we allow the valuation functions to be sensitive to the behavior of each type separately, even when all the valuation functions are continuous. Here, we prove that if there exists a *well-behaved equilibrium*, that is an equilibrium in which everyone with non-negative utility buys on some day, then it is unique. We show this for an infinite number of types in the aggregate model which generalizes both the linear and symmetric models.

Recall that we allow for each buyer $b \in [0, 1]$ to have a unique type in the aggregate model such that the valuation function of buyer b is F_i^b. We will show that in all of the well-behaved equilibrium points the fraction of people buying on each day is the same. In turn, it implies that the revenue of all well-behaved equilibrium points is the same and hence the well-behaved equilibria are revenue-unique. In what follows, we consider the equilibria of a fixed price sequence p. We start with a definition: Consider two well-behaved equilibria x and y. Partition the set of k days to two sets as follows: We call a day i a *level 1* day, and denote it by $i \in D_1(x, y)$, if $X_i < Y_i$. Otherwise, if $X_i \geq Y_i$, we call i a *level 2* day and denote it by $i \in D_2(x, y)$.

Lemma 3 *Assume that there exist two distinct well-behaved equilibria x and y. Then there exists a buyer whose strategy in x is a day i such that $i \in D_1(x, y)$ and whose strategy in y is $j \in D_2(x, y)$.*

Proof : Since $x \neq y$, assume WLOG that there exists a day i so that $X_i < Y_i$. Let S_1 be the set of buyers who have bought on a level 1 day in x and S_2 be the set of buyers who have bought on a level 2 day in y. Further define $S = S_1 \cup S_2$. We show that $|S_1| + |S_2| > |S|$. Therefore S_1 and S_2 have a nonempty intersection whose elements are the buyers we are looking for. We do this by showing that $|S_1| + |S_2| > min(X_{k+1}, Y_{k+1}) \geq |S|$.

First observe that $|S| \leq min(X_{k+1}, Y_{k+1})$ since

- If $b \in S_1$ then b bought on some level 1 day $i \in D_1(x, y)$ in x. Therefore his utility on day i in y is non-negative. As y is a well-behaved equilibrium, this means that b must buy on some day in y. Thus b buys in both x and y.
- If $b \in S_2$ then b bought on some level 2 day in y and so similar reasoning shows that as x is a well-behaved equilibrium, b must also buy on some day in x. Thus again b buys in both x and y.

Hence $|S| \leq min(X_{k+1}, Y_{k+1})$ as claimed.

Next let z_i be equal to x_i for $i \in D_1(x, y)$, and equal to y_i for $i \in D_2(x, y)$.[7] Note that $|S_1| + |S_2| = \sum_{i \in D_1(x,y)} x_i + \sum_{i \in D_2(x,y)} y_i = \sum_{i=1}^{k} z_i$. Thus it suffices to show that $\sum_{i=1}^{k} z_i > |S|$. Let $Z_i = z_1 + \ldots + z_{i-1}$. So we must argue that $Z_{k+1} > min(X_{k+1}, Y_{k+1})$. To do so we use the following claim:

Claim 4 *For each day i if $Z_i \geq$ (respectively $>$) $\min(X_i, Y_i)$, then we have $Z_{i+1} \geq$ (respectively $>$) $\min(X_{i+1}, Y_{i+1})$.*

Proof : For day i, there are four possibilities: If i and $i-1$ are level 1 days, then $Z_{i+1} = Z_i + z_i = Z_i + x_i \geq (>) X_i + x_i = X_{i+1}$. If i is a level 1 day and $i-1$ is a level 2 day, then $Z_{i+1} = Z_i + z_i = Z_i + x_i \geq (>) Y_i + x_i > X_i + x_i = X_{i+1}$. Note that in this case we get strict inequality even assuming weak inequality for i. If i is a level 2 day and $i-1$ is a level 1 day, then $Z_{i+1} = Z_i + z_i = Z_i + y_i \geq (>) X_i + y_i \geq Y_i + y_i = Y_{i+1}$. Finally, if i and $i-1$ are both level 2 days, then $Z_{i+1} = Z_i + z_i = Z_i + y_i \geq (>) Y_i + y_i = Y_{i+1}$. □

Now we complete the proof of lemma by making the following two observations: First, for $i = 1$, we have $X_1 = Y_1 = Z_1 = 0$ so by induction for all i we have $Z_i \geq \min(X_i, Y_i)$. Second, there exists a day of level 1, and since day 1 is level 2, there must be a day i that falls in the second case of the claim for which we must have $Z_i > \min(X_i, Y_i)$. Then by induction we will have $Z_{k+1} > min(X_{k+1}, Y_{k+1})$. □

Theorem 5 *Let $F_i^b(X)$ be a strictly increasing function for each buyer b and day i. For a price sequence p and two well-behaved equilibrium points x and y, we have $X_i = Y_i$, i.e. the fraction of buyers who have bought the product before day i is unique.*

Proof : Assume for contradiction that we have two well-behaved equilibrium points x and y and a day i for which $X_i \neq Y_i$. Again assume without loss of generality that $X_i < Y_i$. By lemma 3 we know that there exists a buyer b who buys on a level 1 day in x and buys on a level 2 day in y. Assume that b buys on day i in x and on day j in y. Then $F_i^b(X_i) - p_i \geq F_j^b(X_j) - p_j$ and $F_j^b(Y_j) - p_j \geq F_i^b(Y_i) - p_i$. Adding the two inequalities we get: $F_i^b(X_i) + F_j^b(Y_j) \geq F_j^b(X_j) + F_i^b(Y_i)$. On the other hand since i is a level 1 day, $X_i < Y_i$; hence by monotonicity $F_i^b(X_i) < F_i^b(Y_i)$. Since j is a level 2 day, $X_j \geq Y_j$; hence $F_j^b(Y_j) \leq F_j^b(X_j)$. The addition of these two inequalities contradicts the previous one. □

[7] Strictly speaking z is not a valid marginal strategy profile as some buyers will buy in two different days in z.

4 Revenue Maximization

In this section, we solve the revenue-maximizing problem in two special cases: the discounted version of the symmetric model, and the general linear model without discount factors. In both cases, we provide an FPTAS to compute the revenue-maximizing price sequence.

4.1 Symmetric Setting

Since all players in this model have the same valuation function F, the marginal strategy profile matrix will reduce to the vector $x = (x_1, \ldots, x_k)$. Also, fixing p and x, the utility of buyer b for the item on day i is $F_i^b(X_i) = F(X_i)\beta^i(1-\alpha)^i - p_i(1-\alpha)^i$, and the revenue $R(p,x) = \sum_i x_i p_i (1-\alpha)^i$. By renaming $q_i = p_i(1-\alpha)^i$ and $\gamma = \beta(1-\alpha)$, the utility of buyer b for the item on day i will be $F(X_i)\gamma^i - q_i$, and the revenue becomes $\sum_i x_i q_i$. Using this new notation, we may assume without loss of generality that the only discount factor is γ. For convenience, we use p for the discounted prices q.

Since we only have one type in this model, we know that the utility of buying in day i is equal among all players. We use the term *utility of a day* i, denoted by u_i, for $u_i = F(X_i)\gamma^i - p_i$. Define $u(p,x) = \max_i u_i$. Consider a price sequence and its equilibrium strategy profile x. We get the following properties immediately from the facts that players are utility maximizing: (i) players are allowed to choose inaction and have utility zero, (ii) they choose to buy if there is a day with a strictly positive utility. First, if there is an i with $x_i > 0$, then $u(p,x) \geq 0$ and $u_i = u(p,x)$. Second, if there is a day i with $x_i > 0$, then $\sum_{i=1}^k x_i = 1$.

First, we observe the following lemma:

Lemma 6 *Let $\hat{p}$ be the revenue-maximizing price vector that results in equilibrium $\hat{x}$. Then $u(\hat{p},\hat{x}) = 0$ (see appendix B)*

Assume that there is a price sequence p with equilibrium x and $u(p,x) = 0$ such that for some day i, we have $x_i = 0$ and $x_{i+1} > 0$. Then we can define a new price sequence $\tilde{p}$ which is equal to p except that $\tilde{p}_j = p_{j+1}/\gamma$ for each $j \geq i$. Also define the vector $\tilde{x}$ to be equal to x except that $\tilde{x}_j = x_{j+1}$ for each $j \geq i$, and $\tilde{x}_k = 0$. One can observe that the pair $(\tilde{p}, \tilde{x})$ is an equilibrium with no less revenue. So we can assume WLOG that for a revenue maximizing price sequence $\hat{p}$ associated with $\hat{x}$, there exists a $k' \leq k$ such that $x_i \neq 0$ if and only if $i \leq k'$. For such a price sequence, lemma 6 shows that $F(X_i)\gamma^i - p_i = 0$ for each $1 \leq i \leq k'$. As a result, we have $X_i = F^{-1}(p_i/\gamma^i)$, which is well-defined as F is increasing. Now set $p'_t = p_t/\gamma^t$. The fraction of people buying on day i and paying price p_i is equal to $x_i = F^{-1}(p'_{i+1}) - F^{-1}(p'_i)$. So the revenue is $\sum_i x_i p_i = \sum_i (F^{-1}(p'_{i+1}) - F^{-1}(p'_i)) p'_i \gamma^i$. This sum is equal to the sum of the areas of a number of rectangles, discounted by γ, that are fit under the graph of F (See figure 1). So the revenue maximization problem reduces to the following *rectangular covering* problem.

Definition 4 Rectangular Covering Problem (RCP) *Given an increasing function F and an integer k, find a sequence p of size at most k that maximizes the discounted total area of the rectangles fit under the graph of F, that is, $p \in \arg\max_{p'} \sum_t (F^{-1}(p'_{t+1}) - F^{-1}(p'_t)) p'_t \gamma^t$.*

In Appendix C.1, we present an FPTAS to solve the rectangular covering problem for concave valuation function and then show how to generalize the proof to non-concave functions (See Theorem 17). Also see appendix C for a general treatment of different versions of the RCP.

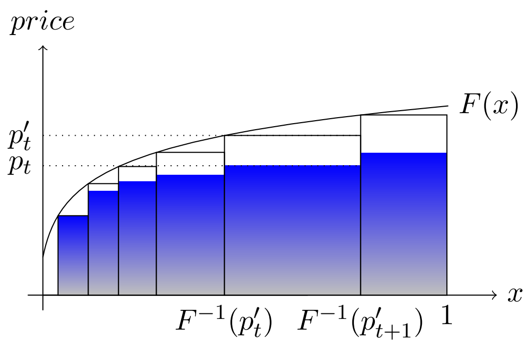

Figure 1: The discounted RCP problem. The blue area is the total area covered by rectangles, discounted by the index of each rectangle).

4.2 Linear Version

To solve the problem in the linear version, we first study the properties of any equilibrium. We also show how to derive prices and the total revenue from vector x. Then we study the revenue maximization problem in linear version in section 4.2.2. We will prove that we can approximate the maximum revenue by solving the Rectangular Covering Problem for a specific curve. This curve is obtained from the function F and the distribution function $\mathcal{C}$.

4.2.1 Equilibrium Properties:

Fix a price sequence p with an equilibrium x. Since we do not have discounts, we can assume without loss that for some $k' \leq k$, a purchase happens in day i if and only if $i \leq k'$ (just remove the days with no purchase to the end). So we can assume that for all $i < j \leq k'$, we have $X_i < X_j$. The utility of a buyer b for buying on day i is $I_b + C_b F(X_i) - p_i$. In order to concentrate on network externalities, for now we restrict the model and assume that I_b is always equal to a fixed constant I for all buyers. Then the utility can be written as $I + C_b F(X_i) - p_i$.

Consider the set of points $q_i = (F(X_i), p_i), 1 \leq i \leq k'$. Define $\hat{s}(i,j)$ to be equal to $(p_i - p_j)/(F(X_i) - F(X_j))$, which is the slope of the line between points q_i and q_j. Also let $s(i) = \hat{s}(i, i+1)$. In lemma 7, we prove that s is non-decreasing in i. Lemma 8 then shows that the utility of buyer b will be maximized on day i if and only if $C_b \in [s(i-1), s(i)]$. Finally, we use these lemmas in Lemma 9 to show how to find a price vector given x. We use these properties in the next section in order to find the desirable equilibrium.

For a fixed b, let $i > j$ be two distinct days. The player prefers day i to j if $I + C_b F(X_i) - p_i \geq I + C_b F(X_j) - p_j$. The above inequality can be written as (recall that we know $X_i > X_j$, and therefore $F(X_i) > F(X_j)$):$C_b \geq \frac{p_i - p_j}{F(X_i) - F(X_j)}$. The converse is also true. If C_b is less than $(p_i - p_j)/(F(X_i) - F(X_j))$, then day j will be preferred to day i.

Lemma 7 *For the function s as defined above, $s(i)$ is non-decreasing in i.*

Proof : Let $i, i+1, i+2$ be three consecutive days, and let b be a buyer who chooses to buy on day $i+1$. For b, day $i+1$ is at least as good as days $i, i+2$. Hence C_b must be greater than or equal to $s(i)$ and less than or equal to $s(i+1)$. We conclude that $s(i+1) \geq s(i)$. □

Lemma 8 *If $C_b \in [s(i-1), s(i)]$ then b will have the maximum utility buying on day i.*

Proof : Assume that $C_b \in [s(i-1), s(i)]$. For each $j > i$, the utility of day i is at least as good as that of day j, because $C_b \le s(i) \le \hat{s}(i,j)$. Similarly for each $j < i$, the utility of day i is at least as good as that of day j, because $C_b \ge s(i-1) \ge \hat{s}(j,i)$. The two special cases $C_b \in [0, s(1)]$ and $C_b \in [s(k-1), \infty)$ are dealt with the same arguments. □

This lemma enables us to find the key relations between prices and vector x.

Lemma 9 *In an equilibrium, the following holds for each* $2 \le i \le k$*:* $p_i - p_{i-1} = (F(X_i) - F(X_{i-1}))\mathcal{C}^{-1}(X_i)$.

Proof : The fraction of people who buy on day i is exactly the fraction whose C_b's lie inside interval $[s(i-1), s(i)]$. So we have $x_i = X_{i+1} - X_i = \mathcal{C}(s(i) - s(i-1))$. The two sequences $\{X_1, \ldots, X_k\}$ and $\{\mathcal{C}(s(0)), \mathcal{C}(s(1)), \ldots, \mathcal{C}(s(k-1))\}$ have identical differences of consecutive terms. They also have identical initial elements $X_1 = \mathcal{C}(s(0)) = 0$. Hence they are identical and we have $X_i = \mathcal{C}(s(i-1))$.

On the other hand $s(i-1)$ is equal to $\frac{p_i - p_{i-1}}{F(X_i) - F(X_{i-1})}$ by definition. Therefore we can conclude the desirable result. □

4.2.2 Revenue Maximization:

We have analyzed the properties of any equilibrium. In this part, we study properties of equilibria which are candidates for the revenue-maximizing one. We show in the revenue-maximizing equilibrium, the first price p_1 will be equal to I in Lemma 10. Using this result, we express total revenue in the revenue-maximizing equilibrium as a function of vector x. Details are in Lemma 11. Finally, in Lemma 12, we prove that the revenue maximization problem can be reduced to Rectangular Covering Problem, and therefore there exists an FPTAS to solve the revenue maximization problem (Theorem 19).

Lemma 10 *In a revenue-maximizing equilibrium,* $p_1 = I$.

Proof : Let x, p be the defining vectors of an equilibrium. Obviously $p_1 \le I$, because people who buy on the first day, have nonnegative utility. Now raise all elements of p by $I - p_1$ to get p'. It's easy to check that x, p' still define an equilibrium. The new equilibrium's revenue is greater than the original one. Hence $p_1 = I$ in the revenue-maximizing equilibrium. □

Note that by calculating all X_i with respect to vector x, we know the values $p_i - p_{i-1}$ using Lemma 9. On the other hand, $p_1 = I$ in the revenue-maximizing equilibrium. So we would know all p_i's. So x uniquely determines prices. Let Price(x) be price vector which has been determined by vector x. It is easy to verify that vectors x and Price(x) make an equilibrium. Hence it suffices to view everything as functions of the free variable x. Now we express the revenue in a candidate equilibrium in terms of vector x.

Lemma 11 *If* x *and* p *correspond to the revenue-maximizing equilibrium, the total revenue can be expressed by the following formula:*

$$R(p,x) = I + \sum_{i=2}^{k} (1 - X_i)\mathcal{C}^{-1}(X_i)(F(X_i) - F(X_{i-1}))$$

Proof : Since the utility of buying on the first day is nonnegative for everybody, all buyers would choose to buy. i.e., $X_{k+1} = 1$. The total revenue can be written as:

$$R(p,x) = \sum_{i=1}^{k} p_i x_i = p_1 \left(\sum_{j=1}^{k} x_j \right) + \sum_{i=2}^{k} (p_i - p_{i-1}) \left(\sum_{j=i}^{k} x_j \right)$$

To interpret the above formula, note that the price change from day $i-1$ to day i is exerted on all buyers who have bought on day i or later. Since $\sum_{j=1}^{k} x_j = 1$, we can rewrite the above formula as $R(p,x) = p_1 + \sum_{i=2}^{k}(p_i - p_{i-1})(1 - X_i)$. Now we can substitute for p_1 and $p_i - p_{i-1}$ from Lemmas 10 and 9 and write revenue as a function of vector x. Therefore we have$R(p,x) = I + \sum_{i=2}^{k}(1 - X_i)\mathcal{C}^{-1}(X_i)(F(X_i) - F(X_{i-1}))$, which is the desired result. □

The next step is to reduce the problem of maximizing revenue to Rectangular Covering Problem. Note that I is a constant in Lemma 11 which does not affect revenue maximization.

Lemma 12 *The problem of maximizing $\sum_{i=2}^{k}(1 - X_i)\mathcal{C}^{-1}(X_i)(F(X_i) - F(X_{i-1}))$ can be reduced to the Rectangular Covering Problem (see appendix D).*

We have proved that the revenue maximization problem can be solved using Rectangular Covering Problem. Therefore, there exists an FPTAS to solve this problem (Theorem 19 in appendix C.2).

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A Examples

A.1 Equilibria May Not Exist

Using an example we show that the market may have no equilibrium point when valuation function is not continuous.

Example 13 *Suppose there are two days $k = 2$ and only one type in the market. Let $F_i(X) = 1$ if $X \leq \frac{1}{2}$ and $F_i(X) = \frac{3}{2} + X$ otherwise, for $i = 1, 2$. We claim price trajectory $p = (0, 1)$ has no equilibrium. Consider any strategy profile X. If $x_1 \leq 1/2$, then the payoff of day 1 is 1 and day 2 is zero, so buyers who don't buy in day one are not playing a best-response so X is not an equilibrium. On the other hand, if $x_1 > 1/2$ then the payoff on day 1 is still 1 but the payoff on day 2 is now greater than $3/2 + 1/2 - 1 \geq 1$. Hence the buyers who buy in day one are not playing a best-response and so X is not an equilibrium.*

While the previous example is enough to show continuity is necessary for existence of equilibria, one might notice that we can resolve the issue by changing the function to $F_i(X) = 1$ if $X < \frac{1}{2}$ and $F_i^t(X) = \frac{3}{2} + X$ otherwise, for $i = 1, 2$. In this case $x = (1/2, 1/2)$ is an equilibrium for price trajectory $p = (0, 1)$. The following example is more robust to the changes in the valuation function:

Example 14 *Assume there are three types and the fraction of each type is $\frac{1}{3}$. The valuation function $F_i^t(X) = 2$ if $X_{t'} < \frac{1}{3}$ and $F_i^t(X) = 4$, where $t' = 2, 3, 1$ respectively for $t = 1, 2, 3$. Then similar reasoning shows price trajectory $p = (1, 2)$ has no equilibrium.*

A.2 Equilibria May Not Be Unique

We show by an example that the market may have more that one equilibrium point with different revenue.

Example 15 *Assume there are two types and $F_i^t(Y) = Y_{t'} + 2$ where $t = 1, 2$ and $t' \neq t$. In other words a social valuation function is only depends on fraction of buyers of another type. The population of type 1 buyers are 0.3 and the population of type 2 buyers are 0.7. Suppose the seller wants to sell the product in two days and $p_1 = 1$ and $p_2 = 1.2$.*

It is clear that two vectors $X = ((0, 0), (0.3, 0))$ and $X' = ((0, 0), (0, 0.7))$ are equilibrium. In the first equilibrium all the type 1(2) buyers have bought the product on day 1(2). The revenue is $0.3 \times 1 + 0.7 \times 1.2 = 1.14$ in this equilibrium. In the second one all the type 1(2) buyers have bought the product on day 2(1). The revenue is $0.7 \times 1 + 0.3 \times 1.2 = 1.06$ in this equilibrium.

B Proof of Lemma 6

Proof : Assume for contradiction that $u(\hat{p}, \hat{x}) = w > 0$. Let $w = (w, \ldots, w)$ be a vector of length k with all of its elements equal to w, and consider the price sequence $\hat{p} + w$ and vector $\hat{x}$. The utility of each day decreases by the same amount w, and the set of maximizers have positive utility, that is, $u(\hat{p} + w, \hat{x}) \geq 0$. Therefore, each day i with $x_i > 0$ is still a maximizer with $u_i \geq 0$. We conclude that $\hat{x}$ is an equilibrium for $\hat{p} + w$. Since $\sum_i x_i = 1$, this price sequence has strictly greater revenue, contradicting the optimality of $\hat{p}$. $\square$

C Rectangular Covering Problem

Definition 5 *Rectangular Covering Problem: A function $F : [0, 1] \to \mathcal{R}$ is given. We want to find k indices $x_1, x_2, \cdots, x_k$ in order to maximize the goal function:*

$$\sum_{i=0}^{k} (x_{i+1} - x_i) F(x_i) \gamma^i$$

Where $x_0 = 0$ and $x_{k+1} = 1$.

In the rectangular covering problem we want to place k rectangles under the curve F, and maximizing the discounted covering area. We propose an FPTAS algorithm for this problem for concave function F in C.1 and for bounded slope function F in C.2. Algorithm will be described in Lemma 16 can be used to solve the rectangular covering problem for general function F with $F(0) > 0$. The running-time of the algorithm is $poly(k, \log_{1+\epsilon} F(1)/F(0))$ in this case. In the remaining part of paper we assume that F is increasing. We prove that this assumption does not hurt the generality in C.3.

Now we define some restricted versions of the rectangular covering problem which are useful in the paper.

Definition 6 *δ-Rectangular Covering Problem: it is an instance of rectangular covering problem in which every $F(x_i)$ should be greater than or equal to δ.*

Definition 7 *(δ_2, δ_1)-Rectangular Covering Problem: it is an instance of rectangular covering problem in which every $F(x_i)$ should be out of a given interval (δ_2, δ_1).*

C.1 Concave Function F

In this section we propose an FPTAS algorithm to solve the rectangular covering problem for concave function F.

Lemma 16 *For every $\epsilon > 0$ and $\delta \geq 0$, the δ-rectangular covering problem can be solved in $poly(k, \log_{1+\epsilon}(F(1)/\delta))$ time, with approximation factor $1 + \epsilon$.*

Proof : At first we define a sequence $S = (s_1, s_2, \cdots, s_m)$ and then we prove that if we choose indices $x_1, x_2, \cdots, x_k$ from sequence S, we could approximate the optimum. Let $s_i = F^{-1}((1 + \epsilon)^{i-1}\delta)$. In fact $F(s_{i+1}) = (1 + \epsilon)F(s_i)$ for every $i < m$, $s_0 = 0$ and $s_m \leq 1 < s_{m+1}$.

Assume optimum indices are $o_1, o_2, \cdots, o_k$ in the rectangular covering problem. The goal function is equal to $O = \sum_{i=0}^{k}(o_{i+1} - o_i)F(o_i)\gamma^i$ in the optimum solution. Let x_i be the maximum index of sequence

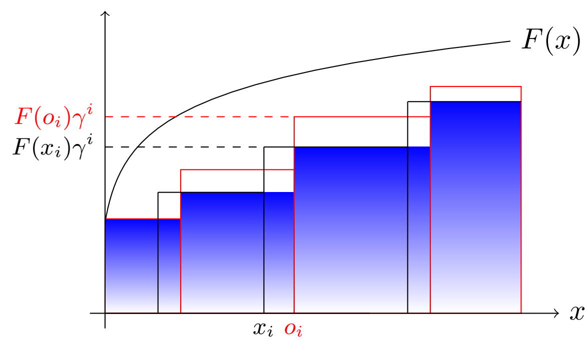

Figure 2: The Region with Area A

S with value no more than o_i. So $F(o_i) \leq F(x_i)(1+\epsilon)$. We can bound O as follows: (In all equations assume $x_0 = o_0 = 0$ and $o_{k+1} = x_{k+1} = 1$)

$$O = \sum_{i=0}^{k}(o_{i+1} - o_i)F(o_i)\gamma^i \leq \sum_{i=0}^{k}(o_{i+1} - o_i)(1+\epsilon)F(x_i)\gamma^i = (1+\epsilon)A \tag{1}$$

The region with area A has been shown in figure 2. It is clear that A is less than or equal to $\sum_{i=0}^{k}(x_{i+1} - x_i)F(x_i)\gamma^i$. So if we choose indices $x_1, x_2, \cdots, x_k$ from sequence S we can approximate the optimum with factor $1+\epsilon$.

Now we design a dynamic programming algorithm to find indices $x_1, x_2, \cdots, x_k$ from S. Let $A[n, r]$ is equal to the best solution when we want to select r indices from sequence $(s_1, s_2, \cdots, s_n)$ and also s_n has been selected. We have:

$$A[n, r] = max_{n'<n}\{A[n', r-1] + (s_n - s_{n'})F(s_{n'})\gamma^r\}$$

The running time of the algorithm is $\Theta(poly(m, k))$, where $m = \Theta(\log_{1+\epsilon}(F(1)/\delta))$ □

Now we are ready to propose an FPTAS algorithm for the rectangular covering problem with concave function F.

Theorem 17 *For every $\epsilon > 0$, the rectangular covering problem with concave function F can be solved in $poly(k, 1/\log(1+\epsilon), \log(1/\epsilon))$ time with approximation factor $1+\epsilon$.*

Proof : Let $\epsilon' = \epsilon/2$. Assume we want to solve the δ-rectangular covering problem for function F with $\delta = F(1)(\frac{1}{1+\epsilon'})^n$, where $n = \log_{1+\epsilon'} \frac{1+2\epsilon'}{4\epsilon'}$. Because F is concave the optimum solution for the rectangular covering problem (OPT) is at least $F(1)/4$. Let the optimum for the δ-rectangular covering problem be OPT_δ and the solution found by Lemma 16 be A_δ. We have proved that $A_\delta \geq \frac{OPT_\delta}{1+\epsilon'}$. On the other hand, the optimum solution for the rectangular covering problem is at most $OPT_\delta + \delta$. So $A_\delta \geq \frac{OPT-\delta}{1+\epsilon'} \geq \frac{OPT}{1+\epsilon'} - \frac{OPT}{4(1+\epsilon')^{n+1}} = \frac{OPT}{1+\epsilon}$.

We have used Lemma 16 with ϵ' and $\delta = F(1)(\frac{1}{1+\epsilon'})^n$. So the algorithm runs in $\Theta(k, \log_{1+\epsilon'} F(1)/\delta) = \Theta(k, n)$ time.

□

C.2 Function F With Bounded Slope

In this section we propose an FPTAS algorithm to solve the rectangular covering problem for function F when $\gamma = 1$ and $F'(x) \leq L$.

Lemma 18 *For every $\delta_1, \delta_2, \epsilon > 0$, the (δ_2, δ_1)-Rectangular Covering Problem can be solved in $poly(k, \log_{1+\epsilon}(\frac{1}{1-F^{-1}(\delta_2)}),$ time, with approximation factor $1+\epsilon$*

Proof : At first we define a sequence $S = (s_1, s_2, \cdots, s_m)$ and then we prove that if we choose indices $x_1, x_2, \cdots, x_k$ from sequence S, we could approximate the optimum. The sequence S consists of two sequences S' and S'', which have been constructed as follows:

- Sequence $S' = (s'_1, s'_2, \cdots, s'_{m'})$ consist of indices with $F(s'_i) \geq \delta_1$. Let $s'_1 = F^{-1}(\delta_1)$ and $s'_{i+1} = F^{-1}((1+\epsilon)(F(s'_i) - \delta_2) + \delta_2)$ and $s'_{m'} = 1$. In fact the we have $F(s'_{i+1}) - \delta_2 \leq (1+\epsilon)(F(s'_i) - \delta_2)$ for every $i < m'$
- Sequence $S'' = (s''_1, s''_2, \cdots, s''_{m''})$ consist of indices with $F(s''_i) \leq \delta_2$. Let $s''_1 = 0$ and $1 - s''_i = (\frac{1}{1+\epsilon})^{i-1}$ and $s''_{m''} = F^{-1}(\delta_2)$. In fact we have $1 - s''_i \leq (1+\epsilon)(1 - s''_{i+1})$.

Assume optimum indices are $o_1, o_2, \cdots, o_k$ in the (δ_2, δ_1)-Rectangular Covering Problem. And let the goal function be $O = \sum_{i=1}^{k}(o_{i+1} - o_i)F(o_i)$ in the optimum solution. We know that every $F(o_i)$ is out of interval (δ_2, δ_1). For every $o_i \leq F^{-}1(\delta_2)$, Let x_i be the minimum index in sequence S (S'') with value not less than o_i, and for every $o_i \geq F^{-}1(\delta_1)$, Let x_i be the maximum index in sequence S (S') with value no more than o_i. Assume $x_i \in S''$ for every index $i < k'$ and $x_i \in S'$ for every index $i \geq k'$. We can bound O as follows: (In all equations assume $x_0 = o_0 = 0$ and $o_{k+1} = x_{k+1} = 1$)

$$O = \sum_{i=0}^{k}(o_{i+1} - o_i)F(o_i) = \sum_{i=0}^{k'-1}(o_{i+1} - o_i)F(o_i) + \sum_{i=k'}^{k}(o_{i+1} - o_i)F(o_i) \tag{2}$$

The right hand side of above equation can be written as:

$$\sum_{i=0}^{k'-1}(o_{i+1} - o_i)F(o_i) = \sum_{i=0}^{k'-1}(1 - o_i)(F(o_i) - F(o_{i-1})) - F(o_{k'-1})(1 - o_{k'}) \tag{3}$$

$$\sum_{i=k'}^{k}(o_{i+1} - o_i)F(o_i) = \sum_{i=k'}^{k}(o_{i+1} - o_i)(F(o_i) - F(o_{k'-1})) + F(o_{k'-1})(1 - o_{k'}) \tag{4}$$

Rewrite equality 2 with respect to equality 3 and 4.

$$O = \sum_{i=0}^{k'-1}(1 - o_i)(F(o_i) - F(o_{i-1})) + \sum_{i=k'}^{k}(o_{i+1} - o_i)(F(o_i) - F(o_{k'-1})) \tag{5}$$

For every $i < k'$ we have $(1 - o_i) \leq (1+\epsilon)(1 - x_i)$. On the other hand $F(x_{i+1}) - \delta_2 \leq (1+\epsilon)(F(x_i) - \delta_2)$, for every $i \geq k'$. Because $F(o_{k'-1}) \leq \delta_2$ we have $F(x_{i+1}) - F(o_{k'-1}) \leq (1+\epsilon)(F(x_i) - F(o_{k'-1}))$, for

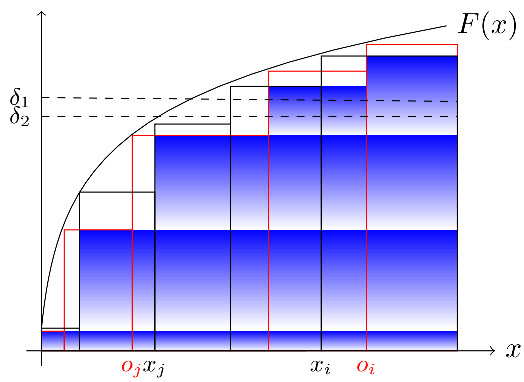

Figure 3: The Region with Area A

every $i \geq k'$. So we have $F(o_i) - F(o_{k'-1}) \leq (1+\epsilon)(F(x_i) - F(o_{k'-1}))$, for every $i \geq k'$. With respect to these facts we have:

$$\begin{aligned} O &\leq (1+\epsilon)\sum_{i=0}^{k'-1}(1-x_i)(F(o_i) - F(o_{i-1})) \\ &+ (1+\epsilon)\sum_{i=k'}^{k}(o_{i+1} - o_i)(F(x_i) - F(o_{k'-1})) \\ &= (1+\epsilon)A \end{aligned} \tag{6}$$

The region with area A has been shown in figure 3. It is clear that A is less than or equal to $\sum_{i=1}^{k}(x_{i+1} - x_i)F(x_i)$. So if we choose indices $x_1, x_2, \cdots, x_k$ from sequence S we can approximate the optimum with factor $1+\epsilon$. Now we design a dynamic programming algorithm to find indices $x_1, x_2, \cdots, x_k$ from S. Let $A[n,t]$ is equal to the best solution when we want to select indices from sequence $(s_1, s_2, \cdots, s_n)$ and also s_n has been selected. We have:

$$A[n,t] = max_{n'<n}\{A[n', t-1] + (s_n - s_{n'})F(s_{n'})\}$$

□

Consider an instance of Rectangular Covering Problem with optimum goal OPT. Construct k^2+1 instances of (δ_2, δ_1)-Rectangular Covering Problem from Rectangular Covering Problem instance. In the i-th instance we set $\delta_2 = \frac{i-1}{k^2+1}$ and $\delta_1 = \frac{i}{k^2+1}$ and assume the value of goal function in the optimum solution of this instance is OPT_i. Assume $o_1, o_2, ..., o_k$ are the optimum indices in the Rectangular Covering Problem instance. It is clear that there exist $1 \leq j \leq k^2+1$ such that every o_i is out of interval $(\frac{j-1}{k^2+1}, \frac{j}{k^2+1})$. Therefore we have $OPT = OPT_j$. Assume we solve all of the k^2+1 instances of (δ_2, δ_1)-Rectangular Covering Problem with Lemma 18. Let A_i is equal to the output of the algorithm for i-th instance. We proved in Lemma 18 that $OPT_i \leq (1+\epsilon)A_i$. So if return $max_i A_i$ as the output for the Rectangular Covering Problem instance, we can approximate the optimum with factor $1+\epsilon$. Now we prove that this algorithm run in polynomial time. In order to prove this fact we should prove that if we set $\delta_2 = \frac{i-1}{k^2+1}$ and $\delta_1 = \frac{i}{k^2+1}$ then $\log_{1+\epsilon}(\frac{1}{1-F^{-1}(\delta_2)}), \log_{1+\epsilon}(\frac{1-\delta_2}{\delta_1-\delta_2})$ are polynomial. First we have $\frac{1-\delta_2}{\delta_1-\delta_2} = k^2+2-i$ which is polynomial with respect to k. On the other hand we have $\delta_2 + \int_{F^{-1}(\delta_2)}^{1} F'(x)dx = 1$. Therefore $\int_{F^{-1}(\delta_2)}^{1} F'(x)dx = 1-\delta_2 \geq \frac{1}{k^2+1}$. If we assume that $F'(x) \leq L$ we can conclude that $1-F^{-1}(\delta_2) \geq \frac{1}{L(k+1)}$. So $\frac{1}{1-F^{-1}(\delta_2)}$ is at most $L(k^2+1)$ which is polynomial with respect to k and L.

Theorem 19 *For every $\epsilon > 0$, the Rectangular Covering Problem with $F'(x) \leq L$ can be solved in $poly(k, \log_{1+\epsilon} k, \log_{1+\epsilon} L)$ time, with approximation factor $1+\epsilon$.*

C.3 Assumptions about F

In this section, we prove that some restrictions can be assumed about function F in the rectangular covering problemWLOG. These restrictions are:

- *F is increasing*: Let $G(x) = \max_{0 \leq y \leq x} F(y)$. We prove that the best rectangular covering of G and F are exactly the same. We prove this statement by showing every optimum solution of rectangular covering problem for F is a solution for G with same objective value and vice versa. First assume $x_1, x_2, \cdots, x_k$ are optimum indices corresponding to some rectangular covering for curve F. We prove that $F(x_i) = G(x_i)$, for every $1 \leq i \leq k$. If it is not the case, there exists an index j which $F(x_j) < G(x_j)$. Let $x_{j'}$ be the smallest index which $G(x_{j'}) = G(x_j)$. It is clear that $F(x_{j'})$ is the biggest value in interval $[0, x_j]$. Now if we replace x_j by $x_{j'}$ in the optimum indices, the change in the objective function will be:

  $$\begin{aligned} & (x_{j'} - x_{j-1})F(x_{j-1})\gamma^{j-1} + (x_{j+1} - x_{j'})F(x_{j'})\gamma^{j} \\ - \quad & (x_j - x_{j-1})F(x_{j-1})\gamma^{j-1} + (x_{j+1} - x_j)F(x_j)\gamma^{j} \end{aligned}$$

  Note that $F(x_{j'}\gamma^j$ is greater that $F(x_j)\gamma^j$ and $F(x_{j-1})\gamma^{j-1}$. Therefore we can conclude that the amount of change is objective function is positive, which is a contradiction with optimality of indices $x_1, x_2, \cdot, x_k$. So indices $x_1, x_2, \cdot, x_k$ is a solution for curve G with the same objective value.

  On the other hand assume $x_1, x_2, \cdot, x_k$ are optimum indices corresponding to some rectangular cover-ingfor curve G. Note that G is non-decreasing. If there are two indices $x_{j'}$ and x_j such that $x_{j'} < x_j$ and $F(x_{j'} = F(x_j)$, then x_j will not appear in any optimum solution of G. If remove indices with this property from the search space only indices x_i with $G(x_i) = F(x_i)$ remain. So indices $x_1, x_2, \cdot, x_k$ is a solution for curve F with the same objective value.

- $F(1) = 1$: Let $F(1) = c \neq 1$. Define $G(x) = \frac{F(x)}{c}$ and solve the problem for G. Assume $x_1, x_2, \cdots, x_k$ are indices corresponding to some rectangular coveringfor curve G. It is clear that:

  $$\sum_{i=0}^{k}(x_{i+1} - x_i)G(x_i)\gamma^i = \frac{1}{c}\sum_{i=0}^{k}(x_{i+1} - x_i)F(x_i)\gamma^i$$

  So we can convert every solution for covering area under curve G to a solution for covering area under curve F and vice versa. So by solving the rectangular covering problemfor curve G we can solve the problem for F.

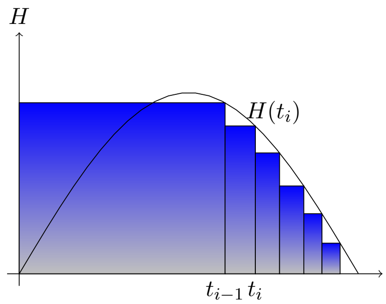

Figure 4: Revenue with respect to non-monotone function H
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D Proof of Lemma 12

Proof : Since F is a monotone and hence bijective function, we can write the revenue as

$$R = \sum_{i=2}^{k}(1 - F^{-1}(F(X_i)))\mathcal{C}^{-1}(F^{-1}(F(X_i))(F(X_i) - F(X_{i-1}))$$

The above formula which only depends on $F(X_i)$'s can be interpreted as a Rectangular Covering Problem instance. Let $F_{min} = F(0), F_{max} = F(1)$. We define $H : [F_{min}, F_{max}] \to \mathbb{R}^{\geq 0}$ as follows : $H(x) = (1-F^{-1}(x))\mathcal{C}^{-1}(F^{-1}(x))$. Note that the revenue is exactly $R = \sum_{i=2}^{k} H(F(X_i))(F(X_i)-F(X_{i-1}))$. Therefore we want to find values $t_i = F(X_i)$'s in order to maximize R. Revenue R can be shown as total area of some rectangles with one corner on curve H (See Figure 4).

The problem of finding these rectangles is very similar to Rectangular Covering Problem. If we define $H'(x) = H(-x)$ and $t'_i = -t_{k-i}$ then we can write the revenue as:

$$\begin{aligned} R &= \sum H(t_i)(t_i - t_{i-1}) = \sum H'(-t_i)(t_i - t_{i-1}) \\ &= \sum H'(t'_{k-i})(t'_{k-i+1} - t'_{k-i}) = \sum H'(t'_j)(t'_{j+1} - t'_j)) \end{aligned}$$

So we should solve Rectangular Covering Problem for function H'. □

E Finding Equilibrium Points

In this section, we propose a method to find an equilibrium profile X for every price vector p in the linear model.

Lemma 20 *Given a price sequence p, we can find an equilibrium profile X.*

Proof : We can safely assume that $p_1, \cdots, p_n$ is a strictly increasing sequence. Because if there is a pair of days i, j for which $i < j$ and $p_i \geq p_j$, no person would buy on day i. If someone buys on day i, day j would have a better influence and not worst price, hence a better utility. So, we can safely remove day i and p_i, which after a number of iterations leads us to a strictly increasing sequence of prices.

Now, we trivially have $X_1 = 0$. Assume that we have evaluated $X_1, X_2, ..., X_{i-1}$ and we want to evaluate X_i.

In lemma 10, we proved that $p_i - p_{i-1} = (F(X_i) - F(X_{i-1}))C^{-1}(X_i)$. Since $F(X_i) - F(X_{i-1})$ and $C^{-1}(X_i)$ are increasing functions of X_i, their product, i.e. $G(X_i) = (F(X_i) - F(X_{i-1}))C^{-1}(X_i)$,

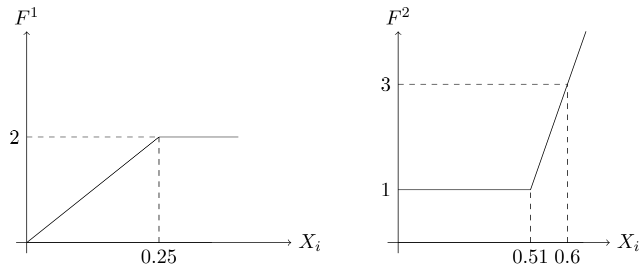

Figure 5: The valuation function of type 1 and 2.

is increasing as well. Hence it has an inverse. Therefore $X_i = G^{-1}(p_i - p_{i-1})$. Note that the equation $G(X_i) = p_i - p_{i-1}$ either has a unique solution or no solution at all. If it has no solution, then no person would have a nonnegative utility buying on day i, and by extension any day after that. Therefore as soon as we reach a no-solution equation, we can stop this phase of the algorithm.

Having obtained the vector $X = (X_1, \cdots, X_k)$ we can find the exact utility of each person buying on each day. For most persons, there will be a unique day with the maximum utility, and hence they will have a unique equilibrium action. Those for whom there are at least two days with the maximum utility, constitute a zero-size fraction of the population, because they're the persons whose C_b lie inside a finite set. This gives us an action profile P. It is unique up to a zero-size fraction of the population.

Since we have a unique equilibrium, and everything we have described so far must apply to that equilibrium, there is no choice but for P to be an equilibrium profile. □

F Existence of Well-Behaved Equilibria

Our revenue results assume the existence of revenue-maximizing well-behaved equilibria for all price sequences. Unfortunately, this does not hold even for the aggregate model, as the following example shows:

Example 21 *Suppose there are three days $k = 3$ and two types in the market, each of which contain half the total population. Let $F_i^1(X_i)$ and $F_i^2(X_i)$ be as shown in Figure 5 for $i = 1, 2, 3$. Note that these are aggregate valuation functions. Consider price trajectory $p = (1, 2, 3)$.*

If we assume that a buyer may decide not to buy the product when the utility of buying is 0, then we have an equilibrium point in this example. The vector $x = ((0, 0.25), (0.35, 0), (0, 0.25))$ is an equilibrium point and a 0.15 fraction of type 1 will not buy the product. However a simple case analysis shows there are no well-behaved equilibria (i.e., equilibria in which a buyer buys when the utility of buying is 0).

Fortunately, we can show that for the linear and symmetric models, well-behaved equilibria exist. We actually show something a bit more general; namely we derive a condition on the utilities that is sufficient to guarantee existence of well-behaved equilibria in which either *all* or *no* buyers buy. We will present the proof for finitely many types; it is clear that it extends to infinitely many types.

Theorem 22 *If either*

$$\min_{t,i} F_i^t(0) \geq \min_i p_i$$

or

$$\max_{t,i} F_i^t(0) < \min_i p_i$$

holds, then there is a well-behaved equilibrium.

Proof : If $\max_{t,i} F_i^t(0) < \min_i p_i$ then it is an equilibrium when no person buys. This is an equilibrium because when no one is buying, utilities are all of the form $F_i^t(0) - p_i$ which is negative. Hence everyone has a negative utility and the strategy profile is a well-behaved equilibrium in which no buyer buys.

So assume that $\min_{t,i} F_i^t(0) \geq \min_i p_i$. Consider the altered price trajectory $(p_1 - \epsilon, \ldots, p_k - \epsilon)$ for any $\epsilon > 0$ (recall our model permits negative prices). Using this price sequence, all buyers buy on some day since the utility of buying on the day with the minimum price is strictly positive. Hence there is a well-behaved equilibrium for this new price trajectory. We will show that this same well-behaved equilibrium is also an equilibrium for the original price trajectory. When prices are all decreased by the same amount all utilities are also decreased by that same amount. Hence the relative ordering of utilities is the same for both price trajectories. So after changing the prices back to the original ones, everyone is still buying an optimal day. The only equilibrium condition that might not hold anymore is the one asserting that buyers only buy when the optimum utility is nonnegative. However, since $\min_{t,i} F_i^t(0) \geq \min_i p_i$, everyone has non-negative utility on the day with the minimum price and so has non-negative optimal utility. Therefore this is a well-behaved equilibrium for the original price trajectory in which all buyers buy. □

Corollary 23 *If $F_i^t(0)$ is the same for all i, t then there is always a well-behaved equilibrium.*

Proof : Since $\max_{t,i} F_i^t(0) = \min_{t,i} F_i^t(0)$, at least one of the conditions of the last theorem holds. □

Note for both the symmetric and linear models, $F_i^t(0)$ is the same for all i, t.