# Graph Clustering using Effective Resistance

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**Abstract**

We design a polynomial time algorithm that for any weighted undirected graph $G = (V, E, \boldsymbol{w})$ and sufficiently large $\delta > 1$, partitions $V$ into subsets $V_1, \ldots, V_h$ for some $h \geq 1$, such that

- at most $\delta^{-1}$ fraction of the weights are between clusters, i.e.
$$w(E - \cup_{i=1}^{h} E(V_i)) \lesssim \frac{w(E)}{\delta};$$
- the effective resistance diameter of each of the induced subgraphs $G[V_i]$ is at most $\delta^3$ times the average weighted degree, i.e.
$$\max_{u,v \in V_i} \mathsf{Reff}_{G[V_i]}(u, v) \lesssim \delta^3 \cdot \frac{|V|}{w(E)} \quad \text{for all } i = 1, \ldots, h.$$

In particular, it is possible to remove one percent of weight of edges of any given graph such that each of the resulting connected components has effective resistance diameter at most the inverse of the average weighted degree.

Our proof is based on a new connection between effective resistance and low conductance sets. We show that if the effective resistance between two vertices $u$ and $v$ is large, then there must be a low conductance cut separating $u$ from $v$. This implies that very mildly expanding graphs have constant effective resistance diameter. We believe that this connection could be of independent interest in algorithm design.

## 1 Introduction

Graph decomposition is a useful algorithmic primitive with various applications. The general framework is to remove few edges so that the remaining components have nice properties, and then specific problems are solved independently in each component. Several types of graph decomposition results have been studied in the literature. The most relevant to this work are low diameter graph decompositions and expander decompositions. We refer the reader to Section 2 for notation and definitions.

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**Low Diameter Graph Decompositions:** Given a weighted undirected graph $G = (V, E, \boldsymbol{w})$ and a parameter $\Delta > 0$, a low diameter graph decomposition algorithm seeks to partition the vertex set $V$ into sets $V_1, \ldots, V_h$ with the following two properties:

- Each component $G[V_i]$ has bounded shortest path diameter, i.e. $\max_{u,v \in V_i} \mathrm{dist}_w(u, v) \leq \Delta$, where $\mathrm{dist}_w(u, v)$ is the shortest path distance between $u$ and $v$ using the edge weight $\boldsymbol{w}$.
- There are not too many edges between the sets $V_i$, i.e. $\left|E - \bigcup_{i=1}^h E(V_i)\right| \leq \frac{D(G)}{\Delta} \cdot |E|$, where $D(G)$ is the "distortion" that depends on the input graph.

This widely studied [LS93, KPR93, Bar96, LS10, AGG+14] primitive (and its generalization to decomposition into padded partitions) has been very useful in designing approximation algorithms [CCC+98, CKR01, FHRT03, FHL08, KR11, BFK+11, LOT14]. This approach is particularly effective when the input graph is of bounded genus $g$ or $K_r$-minor free, in which case $D(G) = \mathcal{O}(\log g)$ [LS10] and $D(G) = \mathcal{O}(r)$ [AGG+14]. For these special graphs, this primitive can be used to proving constant flow-cut gaps [KPR93], proving tight bounds on the Laplacian spectrum [BLR10, KLPT09], and obtaining constant factor approximation algorithms for NP-hard problems [BFK+11, AL17]. However, there are graphs for which $D(G)$ is necessarily $\Omega(\log n)$ where $n$ is the number of vertices, and this translates to a $\Omega(\log n)$ factor loss in applying this approach to general graphs. For example, in a hypercube, if we only delete a small constant fraction of edges, some remaining components will have diameter $\Omega(\log n)$.

**Expander Decompositions:** Given an undirected graph $G = (V, E)$ and a parameter $\phi > 0$, an expander decomposition algorithm seeks to partition the vertex set $V$ into sets $V_1, \ldots, V_h$ with the following two properties.

- Each component $G[V_i]$ is a $\phi$-expander, i.e. $\Phi(G[V_i]) \geq \phi$, where $\Phi(G[V_i])$ is the conductance of the induced subgraph $G[V_i]$; see Section 2 for the definition of conductance.
- There are not too many edges between the sets $V_i$, i.e. $\left|E - \bigcup_{i=1}^h E(V_i)\right| \leq \delta(G, \phi) \cdot |E|$, where $\delta(G, \phi)$ is a parameter depending on the graph $G$ and $\phi$.

This decomposition is also well studied [KVV04, ST11, ABS10, OT14], and is proved useful in solving Laplacian equations, approximating Unique Games, and designing clustering algorithms. It is of natural interest to minimize the parameter $\delta(G, \phi)$. Similar to the low diameter partitioning case, there are graphs where $\delta(G, \phi) \geq \Omega(\phi \cdot \log(n))$. For example, in a hypercube, if we delete a small constant fraction of edges, some remaining components will have conductance $\mathcal{O}(1/\log n)$.

**Motivations:** In some applications, we could not afford to have an $\Omega(\log n)$ factor loss in the approximation ratio. One motivating example is the Unique Games problem. It is known that Unique Games can be solved effectively in graphs with constant conductance [AKK+08] and more generally in graphs with low threshold rank [Kol11, GS11, BRS11], and in graphs with constant diameter [GT06]. Some algorithms for Unique Games on general graphs are based on graph decomposition results that remove a small constant fraction of edges so that the remaining components are of low threshold rank [ABS10] or of low diameter [AL17], but the $\Omega(\log n)$ factor loss in the decomposition is the bottleneck of these algorithms. This leads us to the question of finding a property that is closely related to low diameter and high expansion, so that every graph admits a decomposition into components with such a property without an $\Omega(\log n)$ factor loss.

**Effective Resistance Diameter:** The property that we consider in this paper is having low effective resistance diameter. We interpret the graph $G = (V, E, \boldsymbol{w})$ as an electrical circuit by viewing every edge $e \in E$ as a resistor with resistance $1/w(e)$. The effective resistance distance $\mathsf{Reff}(u, v)$ between the vertices $u$ and $v$ is then the potential difference between $u$ and $v$ when injecting a unit of electric flow into the circuit from the vertex $u$ and removing it out of the circuit from the vertex $v$. We define

$$\mathcal{R}_{diam}(G) := \max_{u,v \in V} \mathsf{Reff}(u, v)$$

as the effective resistance diameter of $G$. Both the properties of low diameter and of high expansion have the property of low effective resistance diameter as a common denominator: The effective resistance distance $\mathsf{Reff}(u, v)$ is upper bounded by the shortest path distance for any graph, and so every low diameter component has low effective resistance diameter. Also, a $d$-regular graph with constant expansion has effective resistance diameter $\mathcal{O}(1/d)$ [BK89, CRR+97], and so an expander graph also has low effective resistance diameter. See Section 2 for more details.

In this paper, we study the connection between effective resistance and graph conductance. Roughly speaking, we show if all sets have mild expansion (see Theorem 1), then the effective resistance diameter is small. We use this observation to design a graph partitioning algorithm to decompose a graph into clusters with effective resistance diameter at most the inverse of the average degree (up to constant losses) while removing only a constant fraction of edges. This shows that although we cannot partition a graph into $\Omega(1)$-expanders by removing a constant fraction of edges, we can partition it into components that satisfy the "electrical properties" of expanders.

**Applications of Effective Resistance:** Besides the motivation from the Unique Games problem, we believe that effective resistance is a natural property to be investigated on its own. The effective resistance distance between two vertices $u, v \in V$ has many useful probabilistic interpretations, such as the commute time [CRR+97], the cover time [Mat88], and the probability of an edge being in a random spanning tree [Kir47]. See Section 2 for more details. Recently, the concept of effective resistance has found surprising applications in spectral sparsification [SS11], in computing maximum flows [CKM+11], in finding thin trees [AO15], and in generating random spanning trees [KM09, MST15, DKP+17]. The recent algorithms in generating a random spanning tree are closely related to our work. Madry and Kelner [KM09] showed how to sample a random spanning tree in time $\widetilde{\mathcal{O}}(m \cdot \sqrt{n})$ where $m$ is the number of edges, faster than the worst case cover time $\widetilde{\mathcal{O}}(m \cdot n)$ (see Section 2). A cruicial ingredient of their algorithm is the low diameter graph decomposition technique, which they use to ensure that the resulting components have small cover time. In subsequent work, Madry, Straszak and Tarnawski [MST15] have improved the time complexity of their algorithm to $\widetilde{\mathcal{O}}(m^{4/3})$ by working with the effective resistance metric instead of the shortest path metric. Indeed, their technique of reducing the effective resistance diameter is similar to our technique – even though it cannot recover our result.

## 1.1 Our Results

Our main technical result is the following connection between effective resistance and graph partitioning.

**Theorem 1.** *Let $G = (V, E)$ be a weighted graph with weights $\boldsymbol{w} \in \mathbb{R}^E_{\geq 0}$. Suppose for any set $S \subseteq V$ with* $\mathrm{vol}(S) \leq \mathrm{vol}(G)/2$ *we have*

$$\Phi(S) \geq \frac{c}{\mathrm{vol}(S)^{1/2-\varepsilon}} \qquad \text{(mild expansion)}$$

*for some $c > 0$ and $1/2 \geq \varepsilon \geq 0$. Then, for any pair of vertices $s, t \in V$, we have*

$$\mathsf{Reff}(s,t) \lesssim \left(\frac{1}{\deg(s)^{2\varepsilon}} + \frac{1}{\deg(t)^{2\varepsilon}}\right) \cdot \frac{1}{\varepsilon \cdot c^2}, \qquad \text{(resistance bound)}$$

*where $\deg(v) = \sum_{u:uv \in E} w(u,v)$ is the weighted degree of $v$.*

In [CRR+97], Chandra et al. proved that a $d$-regular graph with constant expansion has effective resistance diameter $\mathcal{O}(1/d)$. They also proved that the effective resistance diameter of a $d$-dimensional grid is $\mathcal{O}(1/d)$ when $d > 2$ even though it is a poor expander. Theorem 1 can be seen as a common generalization of these two results, using the mild expansion condition as a unifying assumption. Chandra et al. [CRR+97] also showed that the effective resistance diameter of a 2-dimensional grid is $\Theta(\log n)$. Note that for a $\sqrt{n} \times \sqrt{n}$ grid, $\Phi(S) \approx 1/\operatorname{vol}(S)^{1/2}$ for any $k \times k$ square. This shows that the mild expansion assumption of the theorem cannot be weakened in the sense that if $\varepsilon = 0$ for some sets $S$, then $\mathsf{Reff}(s,t)$ may grow as a function of $|V|$.

The proof of Theorem 1 also provides an efficient algorithm to find such a sparse cut. The high-level idea is to prove that if all level sets of the $st$ electric potential vector satisfy the mild expansion condition, then the potential difference between $s$ and $t$ must be small, i.e., $\mathsf{Reff}(s,t)$ is small. Combining with a fast Laplacian solver [ST14], we show that the existence of a pair of vertices $u, v \in V$ with high effective resistance distance implies the existence of a sparse cut which can be found in nearly linear time.

**Corollary 2.** *Let $G = (V, E, \boldsymbol{w})$ be a weighted undirected graph. If $\deg(v) \geq 1/\alpha$ for all $v \in V$, then for any $0 < \varepsilon < 1/2$, there is a subset of vertices $U \subseteq V$ such that*

$$\Phi(U) \lesssim \frac{\alpha^{\varepsilon}}{\sqrt{\mathcal{R}_{diam} \cdot \varepsilon}} \cdot \operatorname{vol}(U)^{\varepsilon - 1/2}.$$

*Furthermore, the set $U$ can be found in time $\widetilde{\mathcal{O}}\left(m \cdot \log\left(\frac{w(E)}{\min_e w(e)}\right)\right)$.*

Using Corollary 2 repeatedly, we can prove the following graph decomposition result.

**Theorem 3** (Main)**.** *Given a weighted undirected graph $G = (V, E, \boldsymbol{w})$, and a large enough parameter $\delta > 1$, there is an algorithm with time complexity $\widetilde{\mathcal{O}}\left(m \cdot n \cdot \log\left(\frac{w(E)}{\min_e w(e)}\right)\right)$ that finds a partition $V = \bigcup_{i=1}^{h} V_i$ satisfying*

$$w\left(E - \bigcup_{i=1}^{h} E(V_i)\right) \lesssim \frac{w(E)}{\delta} \qquad \text{(loss bound)}$$

*and*

$$\mathcal{R}_{diam}(G[V_i]) \lesssim \delta^3 \cdot \frac{n}{w(E)} \qquad \text{(resistance bound)}$$

*for all $i = 1, \ldots, h$.*

Let $G$ be a $d$-regular unweighted graph. Theorem 3 implies that it is possible to remove a constant fraction of the edges of $G$ and decompose $G$ into components with effective resistance diameter at most $\mathcal{O}(1/d)$. Note that $d$-regular $\Omega(1)$-expanders with $\mathcal{R}_{diam} = \mathcal{O}(1/d)$ have the least effective resistance diameter among all $d$-regular graphs. So, even though it is impossible to decompose $d$-regular graphs into graphs with $\Omega(1)$-expansion while removing only a constant fraction of edges, we can find a decomposition with analogous "electrical properties".

We can also view Theorem 3 as a generalization of the following result: Any $d$-regular graph can be decomposed into $\Omega(d)$-edge connected subgraphs by removing only a constant fraction of edges. This is because if the effective resistance diameter of an unweighted graph $G$ is $\epsilon$, then $G$ must be $1/\epsilon$-edge connected. Recall that a graph is $k$-edge connected, if the size of every cut in that graph is at least $k$.

# 2 Preliminaries

In this section, we will first define the notations used in this paper, and then we will review the background in effective resistances, Laplacian solvers, and graph expansions in the following subsections.

Given an undirected graph $G = (V, E)$ and a subset of vertices $U \subseteq V$, we use the notation $E_G(U)$ for the set of edges with both endpoints in $U$, i.e. $E_G(U) = \{\{u, v\} \in E(G) : u, v \in U\}$. We write $U^c$ for the complement of $U$ with respect to $V(G)$, i.e. $U^c = V \backslash U$. The variables $n$ and $m$ stand for the number of vertices and the edges of the graph respectively, i.e. $n = |V|$ and $m = |E|$. We use the notation $\partial_G U$ for the edge boundary of $U \subseteq V$, i.e. $\partial_G U = E_G(U, U^c) = \{\{u, v\} \in E : u \in U, v \in U^c\}$. For a graph $G = (V, E)$ with weights $\boldsymbol{w} \in \mathbb{R}^E_{\geq 0}$, we write $\deg_G(v) = \sum_{u:uv \in E} w(u, v)$ for the weighted degree of $v$. For $S \subseteq V$, the volume $\mathrm{vol}_G(S)$ of $S$ is defined to be $\mathrm{vol}_G(S) = \sum_{s \in S} \deg(s)$. When the graph is clear in the context we may drop the subscript in all aforementioned notation.

Scalar functions and vectors are typed in bold, i.e. $\boldsymbol{x} \in \mathbb{R}^V$, or $\boldsymbol{w} \in \mathbb{R}^E$. For a subset $A \subseteq E$, the notation $w(A)$ stands for the sum of the weights of all edges in $A$, i.e. $w(A) = \sum_{e \in A} w(e)$. The $j$-th canonical basis vector is denoted by $\boldsymbol{e}_j \in \mathbb{R}^V$. Matrices are typed in serif, i.e. $\mathsf{A} \in \mathbb{R}^{V \times V}$.

Time complexities are given in asymptotic notation. We employ the notation $\widetilde{\mathcal{O}}(f(n))$ to hide polylogarithmic factors in $n$, i.e. $\widetilde{\mathcal{O}}(f(n)) = \mathcal{O}(f(n) \cdot \mathrm{polylog}(n))$. We use the notation $f \lesssim g$ for asymptotic inequalities, i.e. $f = \mathcal{O}(g)$; and the notation $f \asymp g$ for asymptotic equalities, i.e. $f = \Theta(g)$.

## 2.1 Electric Flow, Electric Potential, and Effective Resistance

Let $G = (V, E)$ be a given graph with non-negative edge weights $\boldsymbol{w} \in \mathbb{R}^E_{\geq 0}$. The notion of an electric flow arises when one interprets the graph $G$ as an electrical network where every edge $e \in E$ represents a resistor with resistance $1/w(e)$.

We fix an arbitrary orientation $E^{\pm}$ of the edges $E$ and define a unit $st$ flow in this network as a function $\boldsymbol{f} \in \mathbb{R}^{E^{\pm}}_{\geq 0}$ (where for $e \notin E^{\pm}$ we define $f(e) = -f(-e)$) satisfying the following:

$$\sum_{w \in \delta^+(v)} f(vw) - \sum_{u \in \delta^-(v)} f(uv) = \begin{cases} 1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(flow conservation)}$$

where $\delta^+(v)$ is the set of edges having $v$ as the head in our orientation, and $\delta^-(v)$ is the set of edges having $v$ as tail. Let $e = uv \in E^{\pm}$ be an oriented edge. The flow $\boldsymbol{f}$ has to obey Ohm's law

$$f(e) = w(e) \cdot \Delta_e \, \boldsymbol{p} = w(e) \cdot (p(u) - p(v)) \qquad \text{(Ohm's law)}$$

for some vector $\boldsymbol{p} \in \mathbb{R}^V$ which we call the potential vector. The electrical flow between the vertices $s$ and $t$ is the unit $st$ flow that satisfies flow conservation and Ohm's law.

The electrical energy $\mathcal{E}(\boldsymbol{f})$ of a flow $\boldsymbol{f}$ is defined as the following quantity,

$$\mathcal{E}(\boldsymbol{f}) = \sum_{e \in E^{\pm}} \frac{f(e)^2}{w(e)}. \tag{electrical energy}$$

It is known that the electric flow between $s$ and $t$ is the unit $st$ flow with minimal electrical energy. The effective resistance $\mathsf{Reff}(s,t)$ between the vertices $s$ and $t$ is the potential difference between the vertices $s$ and $t$ induced by this flow, i.e. $\mathsf{Reff}(s,t) = p(s) - p(t)$. It is known that the potential difference between $s$ and $t$ equals the energy $\mathcal{E}(\boldsymbol{f}_{st})$ of this flow. This is often referred as Thomson's principle.

The electric potential vector and the effective resistance are known to have the following closed form expressions: Let $\mathsf{W} \in \mathbb{R}^{V \times V}$ be the weighted adjacency matrix of $G$, i.e. the matrix satisfying $W(u,v) = 1[uv \in E] \cdot w(u,v)$, and $\mathsf{D} \in \mathbb{R}^{V \times V}$ the weighted degree matrix, i.e. the diagonal matrix satisfying $D(v,v) = \deg(v) = \sum_{u: uv \in E} w(u,v)$. The (weighted) Laplacian $\mathsf{L}_G \in \mathbb{R}^{V \times V}$ is defined to be the matrix

$$\mathsf{L}_G = \mathsf{D} - \mathsf{W}. \tag{weighted Laplacian}$$

It is well-known that this is a symmetric positive semi-definite matrix. We will take

$$\mathsf{L}_G = \sum_{i=2}^{n} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^{\top}$$

as the spectral decomposition of $\mathsf{L}_G$, where $\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $\mathsf{L}_G$ sorted in increasing order. It is easy to verify $\mathsf{L}_G \mathbf{1} = 0$ and further it can be shown that this is the only vector (up to scaling) satisfying this when $G$ is connected. This means if $G$ is connected, the matrix $\mathsf{L}_G$ is invertible in the subspace perpendicular to $\mathbf{1}$. This inversion will be done by the matrix $\mathsf{L}_G^{\dagger}$, the so-called Moore-Penrose pseudo-inverse of $\mathsf{L}_G$ defined by

$$\mathsf{L}_G^{\dagger} = \sum_{j=2}^{n} \frac{1}{\lambda_j} \boldsymbol{v}_i \boldsymbol{v}_i^{\top}. \tag{pseudo-inverse of $\mathsf{L}_G$}$$

Let $\boldsymbol{f}^{\star} \in \mathbb{R}^{E}$ be the $st$ unit electric flow vector. It can be verified that the $st$ electric potential $\boldsymbol{p}^{\star}$ – i.e. the vector satisfying $w(uv) \cdot (p^{\star}(u) - p^{\star}(v)) = f^{\star}(uv)$ for all $uv \in E^{\pm}$ – satisfies the equation

$$\mathsf{L}_G \boldsymbol{p}^{\star} = \boldsymbol{e}_s - \boldsymbol{e}_t \Longleftrightarrow \boldsymbol{p}^{\star} = \mathsf{L}_G^{\dagger}(\boldsymbol{e}_s - \boldsymbol{e}_t). \tag{2.1}$$

In particular, this implies the following closed form expression for $\mathsf{Reff}(s,t)$

$$\mathsf{Reff}(s,t) = \langle \boldsymbol{e}_s - \boldsymbol{e}_t, \mathsf{L}_G^{\dagger}(\boldsymbol{e}_s - \boldsymbol{e}_t) \rangle. \tag{$st$ effective resistance}$$

It can be verified that this defines a ($\ell_2^2$) metric on the set vertices $V$ of $G$ [KR93], as we have

1. $\mathsf{Reff}(u,v) = 0$ if and only if $u = v$.
2. $\mathsf{Reff}(u,v) = \mathsf{Reff}(v,u)$ for all $u, v \in V$.
3. $\mathsf{Reff}(u,v) + \mathsf{Reff}(v,w) \geq \mathsf{Reff}(u,w)$ for all $u, v, w \in V$.

Further, by routing the unit $st$ flow along the $st$ shortest path we see that the shortest path metric dominates the effective resistance metric, i.e. $\mathsf{Reff}(u, v) \le \operatorname{dist}(u, v)$ for all the pairs of vertices $u, v \in V$.

It is known that the commute time distance $\kappa(u, v)$ between $u$ and $v$ – the expected number of steps a random walk starting from the vertex $u$ needs to visit the vertex $v$ and then return to $u$ – is $\operatorname{vol}(G)$ times the effective resistance distance $\mathsf{Reff}(u, v)$ [CRR+97]. Also, the effective resistance $\mathsf{Reff}(u, v)$ of an edge $uv \in E$ corresponds to the probability of this edge being contained in a uniformly sampled random spanning tree [Kir47]. A well-known result of Matthews [Mat88] relates the effective resistance diameter to the cover time of the graph – the expected number of steps a random walk needs to visit all the vertices of $G$. Aldous [Ald90] and Broder [Bro89] have shown that simulating a random walk until every vertex has been visited allows one to sample a uniformly random spanning tree of the graph.

## 2.2 Solving Laplacian Systems

For our algorithmic results, it will be important to be able to compute electric potentials, and effective resistances quickly. We will do this by appealing to Equation (2.1) and the definition of the *st* effective resistance. Both of these equations require us to solve a Laplacian system. Fortunately, it is known that these systems can be solved in nearly linear time [ST14, KMP10, KMP11, KOSA13, CKM+14, KS16].

**Lemma 4** (The Spielman-Teng Solver, [ST14])**.** *Let a (weighted) Laplacian matrix $\mathsf{L} \in \mathbb{R}^{V \times V}$, a right-hand side vector $\boldsymbol{b} \in \mathbb{R}^V$, and an accuracy parameter $\zeta > 0$ be given. Then, there is a randomized algorithm which takes time $\widetilde{\mathcal{O}}(m \cdot \log(1/\zeta))$ and produces a vector $\widehat{\boldsymbol{x}}$ that satisfies*

$$\left\|\widehat{\boldsymbol{x}} - \mathsf{L}^\dagger \boldsymbol{b}\right\|_{\mathsf{L}} \le \zeta \cdot \|\mathsf{L}^\dagger \boldsymbol{b}\|_{\mathsf{L}} \qquad \text{(accuracy guarantee)}$$

*with constant probability, where $\|\boldsymbol{x}\|_{\mathsf{A}}^2 = \langle \boldsymbol{x}, \mathsf{A}\boldsymbol{x} \rangle$.*

For our purposes it will suffice to pick $\zeta$ inversely polynomial in the size of the graph in the unweighted case, and $1/\operatorname{poly}(w(E)/\min_e w(e), 1/m)$ in the weighted case.

Extending the ideas of Kyng and Sachdeva [KS16], Durfee et al. [DKP+17] show that it is possible to compute approximations for effective resistances between a set of given pairs $S \subseteq V \times V$ efficiently.

**Lemma 5.** *Let $G = (V, E, \boldsymbol{w})$ be a weighted graph, $\beta > 0$ an accuracy parameter, and $S \subseteq V \times V$. There is an $\widetilde{\mathcal{O}}(m + (n + |S|)/\beta^2)$-time algorithm which returns numbers $A(u, v)$ for all $(u, v) \in V$ satisfying*

$$e^{-\beta}\, \mathsf{Reff}(u, v) \le A(u, v) \le e^{\beta}\, \mathsf{Reff}(u, v).$$

This lemma will aid us in computing fast approximations for furthest points in the effective resistance metric. For our purposes, we only need to pick $\beta$ as a small enough constant, i.e. $\beta = \ln(3/2)$. Similar guarantees can also be obtained using the ideas of Spielman and Srivastava [SS11].

## 2.3 Conductance

For a graph $G = (V, E)$ with non-negative edge weights $\boldsymbol{w} \in \mathbb{R}^E_{\ge 0}$, we define the conductance of a set $S \subseteq V$ as

$$\Phi(S) = \frac{w(\partial S)}{\operatorname{vol}(S)}. \qquad \text{(conductance of a set)}$$

The conductance of the graph $G$ is then defined as

$$\Phi(G) = \min\{\Phi(S) : S \subseteq V \text{ and } 2\,\mathrm{vol}(S) \leq \mathrm{vol}(G)\}. \qquad \text{(conductance of a graph)}$$

It is well-known [Che70, AM85] that the conductance of the graph $G$ is controlled by the spectral gap (second smallest eigenvalue) $\tilde{\lambda}_2$ of the normalised Laplacian matrix $\mathsf{D}^{-1/2}\mathsf{L}_G\mathsf{D}^{-1/2}$, i.e.

$$\tilde{\lambda}_2 \lesssim \Phi(G) \lesssim \sqrt{\tilde{\lambda}_2}. \qquad \text{(Cheeger's inequality)}$$

Appealing to the closed form formula for the $st$ effective resistance it can be verified that the spectral gap $\lambda_2$ of the (unnormalised) Laplacian controls the effective resistance distance, i.e.

$$\max_{s,t\in V} \mathsf{Reff}(s,t) \lesssim \frac{1}{\lambda_2}.$$

By an easy application of Cheeger's inequality we see that the expansion controls the effective resistance as well, i.e.

$$\max_{s,t\in V} \mathsf{Reff}(s,t) \lesssim \frac{1}{\Phi(G)^2}.$$

Indeed, Theorem 1 and Corollary 2 will improve upon this bound.

# 3 From Well Separated Points to Sparse Cuts

In this section, we are going to prove Theorem 1 and Corollary 2. As previously mentioned, we will prove that if all the level sets of the potential vector have mild expansion, the effective resistance cannot be high.

**Theorem 1.** *Let $G = (V, E)$ be a weighted graph with weights $\boldsymbol{w} \in \mathbb{R}^E_{\geq 0}$. Suppose for any set $S \subseteq V$ with $\mathrm{vol}(S) \leq \mathrm{vol}(G)/2$ we have*

$$\Phi(S) \geq \frac{c}{\mathrm{vol}(S)^{1/2-\varepsilon}} \qquad \text{(mild expansion)}$$

*for some $c > 0$ and $1/2 \geq \varepsilon \geq 0$. Then, for any pair of vertices $s, t \in V$, we have*

$$\mathsf{Reff}(s,t) \lesssim \left(\frac{1}{\deg(s)^{2\varepsilon}} + \frac{1}{\deg(t)^{2\varepsilon}}\right) \cdot \frac{1}{\varepsilon \cdot c^2}, \qquad \text{(resistance bound)}$$

*where $\deg(v) = \sum_{u:uv\in E} w(u,v)$ is the weighted degree of $v$.*

*Proof.* In the following let $\boldsymbol{f} \in \mathbb{R}^E$ be a unit electric flow from $s$ to $t$, and $\boldsymbol{p} \in \mathbb{R}^V$ be the corresponding vector of potentials where we assume without loss of generality that $p(t) = 0$. We direct our attention to the following threshold sets

$$S_p = \{v \in V : \boldsymbol{p}(v) \geq p\}.$$

Then, we have

$$\sum_{e\in\partial S_p} |f(e)| = 1.$$

Using Ohm's law, we can rewrite this into

$$\sum_{e\in\partial S_p} w(e) \cdot |\Delta_e\, \boldsymbol{p}| = 1, \tag{3.1}$$

where $\Delta_e \boldsymbol{p}$ is the potential difference along the endpoints of the edge $e$. Normalizing this, we get

$$\sum_{e \in \partial S_p} \frac{w(e)}{w(\partial S_p)} \cdot |\Delta_e \boldsymbol{p}| = \frac{1}{w(\partial S_p)}. \tag{3.2}$$

Now, set $\mu(e) = w(e)/w(\partial S_p)$. Restricted over the set of edges $\partial S_p$, $\mu$ is a probability distribution and the LHS of (3.2) corresponds to the expected potential drop when edges $e \in \partial S_p$ are sampled with respect to the probability distribution $\mu$, i.e. we have

$$\mathbb{E}_\mu |\Delta_e \boldsymbol{p}| = \frac{1}{w(\partial S_p)}.$$

Then, by Markov's inequality, we get a set $F \subseteq \partial S_p$ such that

- all edges $f \in F$ satisfy
$$|\Delta_f \boldsymbol{p}| \leq \frac{2}{w(\partial S_p)};$$
- $\mathbb{P}_\mu(e \in F) \geq 1/2$, equivalently
$$w(F) = \sum_{e \in F} w(e) = \sum_{e \in F} w(\partial S_p) \cdot \mu(e) = w(\partial S_p) \cdot \mu(F) \geq \frac{w(\partial S_p)}{2}.$$

Using the observation that the endpoint of an edge $f \in F$ that is not contained in $S_p$ should have potential at least $p - 2/w(\partial S_p)$, we obtain

$$\mathrm{vol}(S_{p-2/w(\partial S_p)}) \geq \mathrm{vol}(S_p) + w(F) \geq \mathrm{vol}(S_p) + \frac{w(\partial S_p)}{2}.$$

Assuming $\mathrm{vol}(\partial S_p) \leq \mathrm{vol}(G)/2$, using the mild expansion property, we have $w(\partial S_p) \geq c\,\mathrm{vol}(S_p)^{1/2+\varepsilon}$. So, from above we get

$$\mathrm{vol}(S_{p-2/c\,\mathrm{vol}(S_p)^{1/2+\varepsilon}}) \geq \mathrm{vol}(S_{p-2/w(\partial S_p)}) \geq \mathrm{vol}(S_p) + \frac{c\,\mathrm{vol}(S_p)^{1/2+\varepsilon}}{2},$$

where in the first inequality we also used that $\mathrm{vol}(S_p)$ increases as $p$ decreases. Now, iterating this procedure $2\,\mathrm{vol}(S_p)^{1/2-\varepsilon}/c$ times we obtain

$$\mathrm{vol}\left(S_{p-\frac{4}{c^2\,\mathrm{vol}(S_p)^{2\varepsilon}}}\right) = \mathrm{vol}\left(S_{p-\frac{2}{c\,\mathrm{vol}(S_p)^{1/2+\varepsilon}} \cdot \frac{2\,\mathrm{vol}(S_p)^{1/2-\varepsilon}}{c}}\right) \geq 2\,\mathrm{vol}(S_p), \tag{3.3}$$

as $\mathrm{vol}(S_p)$ increases as $p$ decreases. We set $p_0 = p(s)$, then $\mathrm{vol}(S_{p_0}) = \deg(s)$. Inductively define

$$p_{k+1} = p_k - \frac{4}{c^2\,\mathrm{vol}(S_{p_k})^{2\varepsilon}}.$$

Then, using the inequality (3.3), we have

$$\mathrm{vol}(S_{p_{k+1}}) \geq 2 \cdot \mathrm{vol}(S_{p_k}). \tag{3.4}$$

Note that we can run the above procedure as long as $\mathrm{vol}(S_p) \leq \mathrm{vol}(G)/2$. Therefore, for some $k^\star \lesssim \log \frac{\mathrm{vol}(G)}{\deg(s)}$, we must have

$$\mathrm{vol}(G) \geq 2 \cdot \mathrm{vol}(S_{p_{k^\star}}) \geq \mathrm{vol}(G)/2.$$

Therefore,

$$p_0 - p_{k^\star} \leq 4 \cdot \sum_{j=0}^{k^\star} \frac{1}{c^2 \operatorname{vol}(S_{p_i})^{2\varepsilon}}.$$

Using (3.4) we get

$$p_0 - p_{k^\star} \lesssim \frac{1}{c^2 \operatorname{vol}(S_0)^{2\varepsilon}} \cdot \sum_{j=0}^{k^\star} \frac{1}{2^{2j\varepsilon}} \lesssim \frac{1}{\deg(s)^{2\varepsilon} \cdot c^2 \cdot \varepsilon},$$

where the last inequality is a geometric sum with ratio $\approx 1/(1+\epsilon)$.

By a similar argument (sending flow from $t$ to $s$), we see that more than half of the vertices have potential smaller than

$$\frac{1}{\deg(t)^{2\varepsilon}} \cdot \frac{1}{\varepsilon \cdot c^2}.$$

Combining these two bounds, we obtain

$$\mathsf{Reff}(s,t) = p(s) \lesssim \left(\frac{1}{\deg(s)^{2\varepsilon}} + \frac{1}{\deg(t)^{2\varepsilon}}\right) \cdot \frac{1}{\varepsilon \cdot c^2},$$

where the equality follows since the flow is a unit flow. □

*Remark* 6. For our proof to go through, we do not need the mild expansion condition to be satisfied by all cuts. It suffices to have this condition satisfied by electric potential threshold cuts $(S_p, S_p^c)$ only.

For computational purposes, it will be important to show that our argument is robust to small perturbations in the potentials, i.e. we need to show that the proof will still go through when we are working with threshold cuts with respect to a vector $\hat{p}$ which is close to the electric potential vector $\boldsymbol{p}$, rather than working with the potential vector $\boldsymbol{p}$ directly. We will show this in Appendix A, Theorem 13.

## 3.1 Finding the Sparse Cuts Algorithmically

Next we prove Corollary 2.

**Corollary 2.** *Let $G = (V, E, \boldsymbol{w})$ be a weighted undirected graph. If $\deg(v) \geq 1/\alpha$ for all $v \in V$, then for any $0 < \varepsilon < 1/2$, there is a subset of vertices $U \subseteq V$ such that*

$$\Phi(U) \lesssim \frac{\alpha^\varepsilon}{\sqrt{\mathcal{R}_{diam} \cdot \varepsilon}} \cdot \operatorname{vol}(U)^{\varepsilon - 1/2}.$$

*Furthermore, the set $U$ can be found in time $\widetilde{\mathcal{O}}\left(m \cdot \log\left(\frac{w(E)}{\min_e w(e)}\right)\right)$.*

*Proof.* First, we prove the existence of $U$. Let $u, v \in V$ such that

$$\mathsf{Reff}(u,v) = \mathcal{R}_{diam}. \tag{3.5}$$

The choice of

$$c \asymp \sqrt{\frac{\frac{1}{\deg(u)^{2\varepsilon}} + \frac{1}{\deg(v)^{2\varepsilon}}}{\mathsf{Reff}(u,v) \cdot \varepsilon}} \quad \text{ensures} \quad \mathsf{Reff}(u,v) > \left(\frac{1}{\deg(s)^{2\varepsilon}} + \frac{1}{\deg(t)^{2\varepsilon}}\right) \cdot \frac{1}{\varepsilon \cdot c^2}. \tag{3.6}$$

Then, by Theorem 1, there must be a threshold set $S_p$ of the potential vector $\boldsymbol{p}$ corresponding to sending one unit of electrical flow from $u$ to $v$ such that

$$\Phi(U) \lesssim \frac{c}{\mathrm{vol}(U)^{1/2-\varepsilon}} \lesssim \frac{\alpha^{\varepsilon}}{\sqrt{\varepsilon \cdot \mathcal{R}_{diam}}} \cdot \mathrm{vol}(U)^{\varepsilon-1/2},$$

where the last inequality follows from our assumption that $\deg(v) \geq 1/\alpha$ for all $v \in V$. This proves the first part of the corollary.

It remains to devise a near linear time algorithm to find the set $U$. First, suppose that we are given the optimum pair of vertices $u, v$ satisfying (3.5). Using the Spielman-Teng solver (Lemma 4), we can compute the potential vector $\boldsymbol{p}$ corresponding to sending one unit of electrical flow from to $u$ to $v$ in time $\widetilde{\mathcal{O}}\Big(m \cdot \log\Big(\frac{w(E)}{\min_e w(e)}\Big)\Big)$. We can then sort the vertices by their potential values in time $\mathcal{O}(n \log n) = \widetilde{\mathcal{O}}(m)$. Finally, we simply go over the sorted list and find the least expanding level set. This can be done in $\mathcal{O}(m)$ time in total, since getting $\partial S_{p(v_i)}$ from $\partial S_{p(v_{i+1})}$ (resp. $\mathrm{vol}(S_{p(v_i)})$ from $\mathrm{vol}(S_{p(v_{i+1})})$) can be done by considering the $\deg(v_i)$ edges $e \in \partial(v_i)$ incident to $v_i$.

It remains to find such an optimal pair of vertices $u, v$ satisfying (3.5). Instead, we find a pair of vertices $u', v'$ such that $\mathsf{Reff}(u', v') \geq \mathcal{R}_{diam}/3$, which is enough for our purposes as this only causes a constant factor loss in the conductance of $U$.

**Lemma 7.** *Let $G$ be a weighted graph. In time $\widetilde{\mathcal{O}}(m)$, one can compute a pair of vertices $u, v \in V$ satisfying*

$$\mathsf{Reff}(u, v) \geq \mathcal{R}_{diam}/3.$$

*Proof.* By the triangle inequality for effective resistances, we have the following inequality for any $u \in V$:

$$\mathcal{R}_{diam} \leq 2 \max_{v \in V} \mathsf{Reff}(u, v). \tag{3.7}$$

Thus, we fix a $u \in V$. Appyling Lemma 5 (with $S = \{u\} \times V$), we get the numbers $A(u, v)$ which multiplicatively approximate $\mathsf{Reff}(u, v)$ within a factor $e^{\beta}$. Let $v^* = \arg\max_{v \in V} A(u, v)$. By combining the inequality (3.7) with

$$\max_{v \in V} \mathsf{Reff}(u, v) \leq e^{\beta} \max_{v \in V} A(u, v) = e^{\beta} A(u, v^*) \leq e^{2\beta} \mathsf{Reff}(u, v^*),$$

we obtain $\mathsf{Reff}(u, v^*) \geq \mathcal{R}_{diam}/3$ for some $\beta = \Theta(1)$. The algorithm consists of an application of Lemma 5 with $|S| = n$, and a linear scan for finding the maximum. Hence, the time bound follows. □

So, Corollary 2 follows by first using Lemma 7 to find $u', v'$ with $\mathsf{Reff}(u', v') \geq R/3$, and then apply Theorem 1 with the choice of $c$ as described in (3.6). □

*Remark* 8. We have avoided treating the issues caused by working with an approximate potential vector for the sake of clarity. This issue is addressed in Appendix A, Corollary 14.

# 4 Low Effective Resistance Diameter Graph Decomposition

In this section we prove Theorem 3.

**Theorem 3** (Main)**.** *Given a weighted undirected graph $G = (V, E, \boldsymbol{w})$, and a large enough parameter $\delta > 1$, there is an algorithm with time complexity $\widetilde{\mathcal{O}}\Big(m \cdot n \cdot \log\Big(\frac{w(E)}{\min_e w(e)}\Big)\Big)$ that finds a partition $V = \bigcup_{i=1}^h V_i$ satisfying*

$$w\left(E - \bigcup_{i=1}^{h} E(V_i)\right) \lesssim \frac{w(E)}{\delta} \qquad \text{(loss bound)}$$

*and*

$$\mathcal{R}_{diam}(G[V_i]) \lesssim \delta^3 \cdot \frac{n}{w(E)} \qquad \text{(resistance bound)}$$

*for all $i = 1, \dots, h$.*

*Proof.* Let $R$ be the target effective resistance diameter and $W$ be the target sum of the weights of edges that we are going to cut. We will write the algorithm in terms of $R, W$, and we will optimize for these parameters later in the proof. Note that $n = |V|$ is the number of vertices of the original graph $G$, and it is fixed throughout the execution of the following algorithm.

**Algorithm 9** (Effective Resistance Partitioning)**.**
**Intput** A graph $H$, and parameters $R, W, n$.
**Output** A partition $\mathcal{P} = \{V_i \mid i = 1, \dots, h\}$ of $V(H)$.

1. If there is a vertex $v \in V(H)$ such that $\deg_H(v) \leq W/(2n)$, then delete all the edges incident to $v$. Repeat this step until there are no such vertices in the remaining graph $H$.
2. Use Lemma 7 to find vertices $u, v$ such that $\mathsf{Reff}(u, v) \geq \mathcal{R}_{diam}(H)/3$.
3. If $\mathsf{Reff}(u, v) \leq R$, return $\{V(H)\}$.
4. Otherwise, find the cut $(U, U^c)$ with $\Phi_H(U) \lesssim \frac{(n/W)^\varepsilon}{\sqrt{\varepsilon \cdot R}} \cdot \mathrm{vol}_H(U)^{\varepsilon - 1/2}$ by invoking Corollary 2, with minimum degree at least $W/(2n)$ and $\varepsilon = 1/4$.
5. Call the algorithm recursively on $H[U]$ and $H[U^c]$.
6. Return the union of the outputs of both recursive calls.

First of all, by construction, every set $V_i$ in the output partition satisfies $\mathcal{R}_{diam}(G[V_i]) \leq 3R$. It is not hard to see that the running time is $\widetilde{\mathcal{O}}(n \cdot m \cdot \log(w(E)/\min_e w(e)))$, as the most expensive of the above algorithm takes time $\widetilde{\mathcal{O}}(m \cdot \log(w(E)/\min_e w(e)))$, and we make at most $n$ recursive calls.

It remains to calculate the sum of the weights of all edges that we cut. Note that we cut edges either when a vertex has a low degree or when we find a low conductance set $U$. We classify the cut edges into two types as follows:

i) Edges $e$ where $e$ is cut as an incident edge of a vertex $v$ with $\deg_H(v) \leq W/2n$.

ii) The rest of the edges, i.e., edges $e$ where $e \in \partial_H(U)$ for some $U$ where $\Phi_H(U) \lesssim \frac{(n/W)^\varepsilon}{\sqrt{\varepsilon \cdot R}} \mathrm{vol}_H(U)^{\varepsilon - 1/2}$.

We observe that we are going to remove edges of type (i) for at most $n$ times, because each such removal isolates a vertex of $G$. So, the sum of the weights of edges of type (i) that we cut is at most $n \cdot W/2n \leq W/2$. It remains to bound the sum of the weight of edges of type (ii) that we cut.

We use an amortization argument: Let $\Psi(e)$ stand for the tokens charged from an edge. We assume that for each edge $e \in E$, the number of tokens $\Psi(e)$ is initially set to 0. Every time we make a cut of type (ii), we assume without loss of generality that $\mathrm{vol}_H(U) \leq \mathrm{vol}(H)/2$ and we modify the number of tokens as follows

$$\Psi(e) := \begin{cases} \Psi(e) + \frac{w(\partial_H U)}{w(E_H(U))} & \text{if } e \in E_H(U) \\ \Psi(e) & \text{otherwise.} \end{cases} \tag{4.1}$$

By definition, after the termination of the algorithm, we have

$$w(\text{set of cut edges of type (ii)}) = \sum_{e \in E} \Psi(e) \cdot w(e). \tag{4.2}$$

Therefore, to bound the total weight of type (ii) edges that are cut, it is enough to show that no edge is charged with too many tokens provided $R$ is large enough.

**Claim 10.** *If $R \gtrsim n/(\varepsilon W)$, we will have $\Psi(e) \lesssim \frac{4}{\sqrt{R \cdot W/8n} - 1}$ for all edges $e \in E$ after the termination of the algorithm.*

*Proof.* Fix an edge $e \in E$. Let $\Delta\, \Psi(e)$ be the increment of $\Psi(e)$ due to a cut $(U, U^c)$. We have

$$\Delta\, \Psi(e) = \frac{w(\partial_H U)}{w(E_H(U))} = 2 \cdot \frac{w(\partial_H U)}{\mathrm{vol}_H(U) - w(\partial_H U)} = 2 \cdot \frac{1}{\frac{1}{\Phi_H(U)} - 1} \lesssim \frac{2c}{\mathrm{vol}_H(U)^{1/2-\varepsilon} - c}, \tag{4.3}$$

where $c$ is chosen as in (3.6) in the proof of Corollary 2 so that $\Phi(U) \leq c/\,\mathrm{vol}(U)^{1/2-\varepsilon}$ for the last inequality to hold. Since the minimum degree is at least $W/2n$ by Step (1) of the algorithm, we have

$$c \asymp \frac{(2n/W)^{\varepsilon}}{\sqrt{\varepsilon \cdot R}}.$$

The minimum degree condition also implies that $\mathrm{vol}_H(U) \geq W/(2n)$. Note that the denominator of the rightmost term of (4.3) is non-negative as long as $\mathrm{vol}_H(U)^{1/2-\epsilon} \geq (W/2n)^{1/2-\varepsilon} \geq c$, which holds when $R \gtrsim n/(\varepsilon W)$.

Let $U_0 \subseteq V(H_0)$ be the set for which $e$ was charged for the last time, and in general $U_k \subseteq V(H_k)$ be the $k$-th last set for which $e$ was charged. We write $\Delta_k \Psi(e)$ to denote the increment in $\Psi(e)$ due to $U_k$.

Note that by (4.1) we have $e \in E_{H_i}(U_i)$ for all $i$. Furthermore, since $\mathrm{vol}_{H_i}(U_i) \leq \mathrm{vol}(H_i)/2 \leq \mathrm{vol}_{H_{i+1}}(U_{i+1})/2$ for all $i$, we have

$$\mathrm{vol}_{H_k}(U_k) \geq 2^k\, \mathrm{vol}_{H_0}(U_0) \tag{4.4}$$

for all $k \geq 0$. Therefore, using (4.3) and (4.4), we can write

$$\begin{aligned} \Psi(e) = \sum_{k \geq 0} \Delta_k\, \Psi(e) \quad &\leq \quad \sum_{k \geq 0} \frac{2c}{\mathrm{vol}_{H_i}(U_i)^{1/2-\varepsilon} - c} \\ &\leq \quad \sum_{k \geq 0} \frac{2c}{(2^k\, \mathrm{vol}_{H_0}(U_0))^{1/2-\varepsilon} - c} \\ &\leq \quad \frac{2c}{\mathrm{vol}_{H_0}(U_0)^{1/2-\varepsilon} - c} \cdot \sum_{k \geq 0} \frac{1}{(2^{1/2-\varepsilon})^k}, \end{aligned}$$

where the last inequality assumes that $\varepsilon < 1/2$. As argued before, the minimum degree condition implies that every vertex is of degree at least $W/2n$ and thus $\mathrm{vol}_{H_0}(U_0) \geq W/(2n)$. Therefore, by the geometric sum formula, we have

$$\Psi(e) \leq \frac{2}{\frac{1}{c}(W/2n)^{1/2-\varepsilon} - 1} \cdot \frac{1}{1 - 2^{\varepsilon - 1/2}}.$$

Plugging the value of $c$ and setting $\varepsilon = 1/4 < 1/2$, we conclude that

$$\Psi(e) \lesssim \frac{2}{\sqrt{\varepsilon \cdot R \cdot W/2n} - 1} \leq \frac{4}{\sqrt{R \cdot W/8n} - 1}.$$

□

Setting $R \asymp \delta^2 \cdot n/W$ for a sufficiently large $\delta^2 > 1$ so that the assumption of Claim 10 is satisfied, it follows from (4.2) that the sum of the weights of all cut edges is at most

$$W/2 + \sum_e \Psi(e) \cdot w(e) \lesssim W/2 + \frac{w(E)}{\delta}.$$

Setting $W = w(E)/\delta$ proves the theorem. This completes the proof of Theorem 3. □

# 5 Conclusions and Open Problems

We have shown that we can decompose a graph into components of bounded effective resistance diameter while losing only a small number of edges. There are few questions which arise naturally from this work.

1. Can the decomposition in Theorem 3 be computed in near linear time? Is this decomposition useful in generating a random spanning tree?
2. For the Unique Games Conjecture, Theorem 3 implies that we can restrict our attention to graphs with bounded effective resistance diameter. Can we solve Unique Games instances better in such graphs? More generally, are there some natural and nontrivial problems that can be solved effectively in graphs of bounded effective resistance diameter?
3. Is there a generalization of Theorem 1 to multi-partitioning, i.e. does the existence of $k$-vertices with high pairwise effective resistance distance help us in finding a $k$-partitioning of the graph where every cut is very sparse?
4. Theorem 1 says that a small-set expander has bounded effective resistance diameter. Is it possible to strengthen Theorem 3 to show that every graph can be decomposed into small-set expanders? This may be used to show that the Small-Set Expansion Conjecture and the Unique Games Conjecture are equivalent, depending on the quantitative bounds.


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# A Robustness of the Proof of Theorem 1

We avoided the issue of picking the accuracy parameter $\epsilon > 0$ for the Laplacian solver we used in Corollary 2. Here, we want to show that the proof is robust enough to small perturbations in the potential vector, i.e. using a Laplacian solver to estimate $s$-$t$ potential vector $\boldsymbol{p} \in \mathbb{R}^V$ by the vector $\hat{\boldsymbol{p}}$, additively within an accuracy of $\eta$, we can still recover our sparse cut.

We first start by noting that $|p(v) - \hat{p}(v)| \leq \eta$ is implied by the stronger inequality,

$$\|\hat{\boldsymbol{p}} - \boldsymbol{p}\|^2 \leq \eta^2. \tag{A.1}$$

We will show that if $\eta$ is polynomially small in the input data, we can still find a sparse cut. Our plan is as follows.

- We will figure out how small we should set the Laplacian solver accuracy $\epsilon$ to ensure (A.1) (Lemma 12).
- We will show that using the mild-expansion of the threshold sets $T_{\hat{p}}$ of the vector $\hat{\boldsymbol{p}}$, we can still prove upper bounds on the effective resistance (Theorem 13).
- Analogously to Corollary 2, we will show that by way of contraposition the existence of a pair with large effective resistance distance means one of the threshold sets $T_{\hat{p}}$ does not satisfy the mild expansion property (Corollary 14).

## A.1 Eigenvalue Bound

We start with a simple eigenvalue bound that will be used to bound the accurancy needed.

**Claim 11.** *For any connected weighted graph $G = (V, E, \boldsymbol{w})$, we have*

$$\lambda_2(G) \gtrsim \min_e w(e) \cdot \left(\frac{\min_e w(e)}{w(E)}\right)^2. \tag{A.2}$$

*Proof.* For any connected weighted graph $G = (V, E, \boldsymbol{w})$, we have the following conductance bound,

$$\min\left\{\frac{|\partial(S)|}{\mathrm{vol}(S)} : \mathrm{vol}(S) \leq \mathrm{vol}(G)/2\right\} = \Phi(G) \geq \frac{\min_e w(e)}{w(E)},$$

which implies

$$\frac{\min_e w(e)}{w(E)} \lesssim \sqrt{\tilde{\lambda}_2(G)} \iff \tilde{\lambda}_2(G) \gtrsim \left(\frac{\min_e w(e)}{w(E)}\right)^2$$

by Cheeger's inequality. Note that

$$\lambda_2(G) = \min_{\boldsymbol{u} \perp \mathbf{1}} \frac{\langle \boldsymbol{u}, \mathsf{L}\boldsymbol{u}\rangle}{\langle \boldsymbol{u}, \boldsymbol{u}\rangle} = \min_{\boldsymbol{v} \perp \mathsf{D}^{1/2}\mathbf{1}} \frac{\langle \mathsf{D}^{-1/2}\boldsymbol{v}, \mathsf{L}\mathsf{D}^{-1/2}\boldsymbol{v}\rangle}{\langle \mathsf{D}^{-1/2}\boldsymbol{v}, \mathsf{D}^{-1/2}\boldsymbol{v}\rangle} \tag{A.3}$$

where the last equality follows by a change of variables $\boldsymbol{u} = \mathsf{D}^{-1/2}\boldsymbol{v}$. This implies that

$$\lambda_2(G) = \min_{\boldsymbol{v} \perp \mathsf{D}^{1/2}\mathbf{1}} \frac{\langle \boldsymbol{v}, \tilde{\mathsf{L}}\boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \mathsf{D}^{-1}\boldsymbol{v} \rangle} \geq \min_{\boldsymbol{v} \perp \mathsf{D}^{1/2}\mathbf{1}} \frac{\langle \boldsymbol{v}, \tilde{\mathsf{L}}\boldsymbol{v} \rangle}{\|\mathsf{D}^{-1}\| \cdot \langle \boldsymbol{v}, \boldsymbol{v} \rangle} \geq \frac{1}{\|\mathsf{D}^{-1}\|} \cdot \tilde{\lambda}_2(G). \tag{A.4}$$

Using $\|\mathsf{D}^{-1}\|^{-1} = \min_v \deg(v) \geq \min_e w(e)$, we obtain $\lambda_2(G) \geq \min_e w(e) \cdot \tilde{\lambda}_2(G)$. Combining everything, we get

$$\lambda_2(G) \gtrsim \min_e w(e) \cdot \left( \frac{\min_e w(e)}{w(E)} \right)^2 \tag{A.5}$$

hence proving the claim. □

## A.2 Picking the Laplacian Solver Accuracy

For $\boldsymbol{b} = \boldsymbol{e}_s - \boldsymbol{e}_t$, the Spielman Teng Solver in Lemma 4 produces a vector $\hat{\boldsymbol{p}}$ such that

$$\|\hat{\boldsymbol{p}} - \mathsf{L}^\dagger \boldsymbol{b}\|_{\mathsf{L}} \leq \zeta \cdot \|\mathsf{L}^\dagger \boldsymbol{b}\|_L.$$

Letting $\boldsymbol{p}$ be the $s$-$t$ electric potential vector, this becomes

$$\|\hat{\boldsymbol{p}} - \boldsymbol{p}\|_{\mathsf{L}} \leq \zeta \cdot \|\boldsymbol{p}\|_{\mathsf{L}}.$$

Using the definition of the $\mathsf{L}$-norm, we have

$$\langle \hat{\boldsymbol{p}} - \boldsymbol{p}, \mathsf{L}(\hat{\boldsymbol{p}} - \boldsymbol{p}) \rangle \leq \zeta^2 \langle \boldsymbol{p}, \mathsf{L}\boldsymbol{p} \rangle = \zeta^2 \cdot \mathsf{Reff}(s,t).$$

Since we are working on the space orthogonal to the nullspace, both $\hat{\boldsymbol{p}}, \boldsymbol{p} \perp \mathbf{1}$ and thus $(\hat{\boldsymbol{p}} - \boldsymbol{p}) \perp \mathbf{1}$. It follows from the definition of $\lambda_2(G)$ that

$$\lambda_2(G) \cdot \|\hat{\boldsymbol{p}} - \boldsymbol{p}\|^2 \leq \zeta^2 \cdot \mathsf{Reff}(s,t).$$

By the eigenvalue bound in (A.2), we have

$$\|\hat{\boldsymbol{p}} - \boldsymbol{p}\|^2 \lesssim \zeta^2 \cdot \mathsf{Reff}(s,t) \cdot \frac{w(E)^2}{(\min_e w(e))^3}.$$

Using the trivial bound $\mathsf{Reff}(s,t) \leq \frac{m}{\min_e w(e)}$ in a connected graph, we get

$$\|\hat{\boldsymbol{p}} - \boldsymbol{p}\|^2 \lesssim \zeta^2 \cdot m \cdot \frac{w(E)^2}{(\min_e w(e))^4}.$$

Therefore, we can set

$$\zeta \asymp \frac{\eta \cdot (\min_e w(e))^2}{w(E) \cdot \sqrt{m}}$$

to get the desired accuracy in (A.1). The above argument is summarized in the following lemma.

**Lemma 12.** *Given a connected weighted graph $G = (V, E, \boldsymbol{w})$, it is possible to compute an estimate $\hat{\boldsymbol{p}}$ of the $s$-$t$ electric potential vector $\boldsymbol{p}$ within an additive accuracy of $\eta$ using the Spielman-Teng solver with accuracy*

$$\zeta \asymp \frac{\eta \cdot (\min_e w(e))^2}{w(E) \cdot \sqrt{m}}.$$

### A.3 How Small Should We Pick $\eta$?

In the proof of Theorem 1, we used the actual potential vector to bound the effective resistance. This is too expensive for algorithmic purposes. We now show that we can use the estimate $\hat{\boldsymbol{p}}$ and the potential sets

$$T_{\hat{p}} = \{v \in V : \hat{p}(v) \geq \hat{p}\}$$

at a small cost. We will show that the mild-expansion of these cuts allows us to bound the effective resistance from above, just as in Theorem 1.

**Theorem 13.** *Let $\hat{\boldsymbol{p}}$ be an additive $\eta$-approximation of the electric potential vector $\boldsymbol{p}$ between $s$ and $t$, i.e.*

$$|\hat{p}(u) - p(u)| \leq \eta \quad \forall u \in V.$$

*If all the threshold cuts $T_{\hat{p}}$ satisfy the mild expansion condition,*

$$w(\partial T_{\hat{p}}) \geq c \cdot \mathrm{vol}(T_{\hat{p}})^{1/2+\varepsilon}$$

*Then, we have*

$$\mathsf{Reff}(s,t) \lesssim \frac{1}{\varepsilon c^2 \deg(s)^{2\varepsilon}} + \frac{1}{\varepsilon c^2 \deg(t)^{2\varepsilon}} + \frac{\eta(48m^{1/2-\varepsilon}\log n + 2c)}{c}.$$

*Proof.* The proof will be very similar to that of Theorem 1, we will just highlight the differences and carry out the relevant computations.

To follow the proof of Theorem 1, we need an upper bound on the quantity

$$\sum_{e \in \partial T_{\hat{p}}} w(e) \cdot |\Delta_e \hat{\boldsymbol{p}}| \leq \sum_{e \in \partial T_{\hat{p}}} w(e) \cdot (|\Delta_e \boldsymbol{p}| + 2\eta) = \sum_{e \in \partial T_{\hat{p}}} w(e) \cdot |\Delta_e \boldsymbol{p}| + 2\eta \cdot w(\partial T_{\hat{p}}). \tag{A.6}$$

where for the first inequality we have used the triangle inequality

$$|\Delta_e \hat{\boldsymbol{p}}| = |\hat{p}(e^+) - \hat{p}(e^-)| \leq |p(e^+) - p(e^-)| + |p(e^+) - \hat{p}(e^+)| + |p(e^-) - \hat{p}(e^-)| \leq |\Delta_e \boldsymbol{p}| + 2\eta.$$

Bounding the RHS of (A.6) is certainly possible by bounding

$$\sum_{e \in \partial T_{\hat{p}}} w(e) \cdot |\Delta_e \hat{\boldsymbol{p}}|. \tag{A.7}$$

In Theorem 1, we used the Equation (3.1) to bound the analogous term, i.e.

$$\sum_{e \in \partial S_p} |f(e)| = \sum_{e \in \partial S_p} w(e) \cdot |\Delta_e \boldsymbol{p}| = 1.$$

We were allowed to do this because $S_p$ is a threshold set, i.e. the $st$ electric flow flows in one direction: from $S_p$ to $S_p^c$. This means that flow conservation insures $\sum_{e \in \partial(S_p)} |f(e)| = 1$, as $s$ has a flow deficit of a unit, and $t$ has a flow surplus of a unit. This is no longer true for $T_{\hat{p}}$ since $T_{\hat{p}}$ is no longer a threshold set of the true potential vector $\boldsymbol{p}$, i.e. we do not necessarily have $\sum_{e \in \partial(T_{\hat{p}})} |f(e)| = 1$.

Before we go on further, we adopt the following convention of taking $uv \in \partial T_{\hat{p}}$ to be an edge with $u \in T_{\hat{p}}$ and $v \notin T_{\hat{p}}$.

In our case the conservation of flow still implies,

$$\sum_{e \in \partial T_{\hat{p}}} f(e) = 1. \tag{A.8}$$

We will take $P^-$ to be the set of edges $uv \in \partial(T_{\hat{p}})$ with $p(u) < p(v)$, and $P^+$ to be the set of edges $uv \in \partial(T_{\hat{p}})$ with $p(u) \geq p(v)$.

Now, note that (A.8) rewrites into,

$$\sum_{uv \in P^+} w(uv) \cdot |\Delta_{uv}\, \boldsymbol{p}| - \sum_{uv \in P^-} w(uv) \cdot |\Delta_{uv}\, \boldsymbol{p}| = 1.$$

In particular, we can manipulate this to obtain

$$\sum_{uv \in \partial T_{\hat{p}}} w(uv) \cdot |\Delta_{uv}\, \boldsymbol{p}| = \sum_{uv \in P^-} w(u,v) \cdot |\Delta_{uv}\, \boldsymbol{p}| + \sum_{uv \in P^-} w(u,v) \cdot |\Delta_{uv}\, \boldsymbol{p}| = 1 + 2 \sum_{uv \in P^-} w(u,v) \cdot |\Delta_{uv}\, \boldsymbol{p}|. \tag{A.9}$$

We see now, proving an upper bound on (A.6) boils down to upper bounding $|\Delta_{uv}\, \boldsymbol{p}|$ for $uv \in P^-$. This can be done by noting $\hat{p}(u) > \hat{p}(v)$ (as $uv \in \partial(T_{\hat{p}})$) and $\boldsymbol{p} \approx \hat{\boldsymbol{p}}$. Formally, we have

$$p(u) + \eta \geq \hat{p}(u) \geq \hat{p} > \hat{p}(v) \geq p(v) - \eta,$$

which means, we must either have $p(u) \geq p(v)$ or it must be the case that we have $p(u) < p(v)$ and $|\Delta_{uv}\, \boldsymbol{p}| \leq 2\eta$. This readily implies the following inequality,

$$\sum_{e \in P^-} w(e) \cdot |\Delta_e\, \boldsymbol{p}| \leq 2\eta \cdot w(\partial T_p). \tag{A.10}$$

Plugging this in (A.9), we get

$$\sum_{e \in \partial T_{\hat{p}}} w(e) \cdot |\Delta_e\, \boldsymbol{p}| = \sum_{e \in P^+} w(e) \cdot |\Delta_e\, \boldsymbol{p}| + \sum_{e \in P^-} w(e) \cdot |\Delta_e\, \boldsymbol{p}| \leq 1 + 4\eta \cdot w(\partial T_{\hat{p}}).$$

Combining this with (A.6) yields,

$$\sum_{e \in \partial T_{\hat{p}}} w(e) \cdot |\Delta_e\, \hat{\boldsymbol{p}}| \leq 1 + 6\eta \cdot w(\partial T_{\hat{p}}). \tag{A.11}$$

With this bound, we can proceed as in the proof of Theorem 1. Normalizing (A.11) we obtain,

$$\sum_{e \in \partial T_{\hat{p}}} \frac{w(e)}{w(\partial T_{\hat{p}})} \cdot |\Delta_e\, \hat{\boldsymbol{p}}| \leq \frac{1}{w(\partial T_{\hat{p}})} + 6\eta.$$

Since $\mu(e) = w(e)/w(\partial T_{\hat{p}})$ is a probability distribution, analogously as in the proof of Theorem 1, we obtain a set $F \subseteq \partial T_{\hat{p}}$ by Markov's inequality such that

- all edges $f \in F$ satisfy $|\Delta_f\, \hat{\boldsymbol{p}}| \leq \frac{2}{w(\partial T_{\hat{p}})} + 12\eta$
- the set $F$ is "large", i.e. $w(F) = \mu(F) \cdot w(\partial T_{\hat{p}}) \geq \frac{1}{2} \cdot w(\partial T_{\hat{p}})$.

Now, analogously as in the proof of Theorem 1, we get

$$\mathrm{vol}(T_{\hat{p}-\frac{2}{w(\partial T_{\hat{p}})}-12\eta}) \geq \mathrm{vol}(T_{\hat{p}}) + \frac{1}{2} \cdot w(\partial T_{\hat{p}}).$$

Using mild expansion of $T_{\hat{p}}$, this implies

$$\mathrm{vol}\left(T_{\hat{p}-\frac{2}{c\,\mathrm{vol}(T_{\hat{p}})^{1/2+\varepsilon}}-12\eta}\right) \geq \mathrm{vol}(T_{\hat{p}}) + \frac{c\,\mathrm{vol}(T_{\hat{p}})^{1/2+\varepsilon}}{2}.$$

Iterating this for $(\frac{2}{c}\,\mathrm{vol}(T_{\hat{p}})^{1/2-\varepsilon})$-times, we obtain

$$\mathrm{vol}\left(T_{\hat{p}-\frac{2}{c^2\,\mathrm{vol}(T_{\hat{p}})^{2\varepsilon}}-\frac{24\eta\cdot\mathrm{vol}(T_{\hat{p}})^{1/2-\varepsilon}}{c}}\right) \geq 2\,\mathrm{vol}(T_{\hat{p}}). \tag{A.12}$$

Now, similarly we set $\hat{p}_0 = \hat{p}(s)$ and inductively extend this to $\hat{p}_k$ for $k > 0$ by

$$\hat{p}_{k+1} = \hat{p}_k - \frac{2}{c^2\,\mathrm{vol}(T_{\hat{p}})^{2\varepsilon}} - \frac{24\eta\cdot\mathrm{vol}(T_{\hat{p}})^{1/2-\varepsilon}}{c}.$$

Since $s \in T_{\hat{p}_0}$, we know $\mathrm{vol}(T_{\hat{p}_0}) \geq \deg(s)$, and by (A.12) $\mathrm{vol}(T_{\hat{p}_k}) \geq 2^k \deg(s)$. There exists some $k^\star \lesssim \log n$ satisfying,

$$\mathrm{vol}(G) \geq 2\,\mathrm{vol}(T_{\hat{p}_{k^\star}}) \geq \mathrm{vol}(G)/2. \tag{A.13}$$

It follows that

$$\hat{p}_{k^\star} - \hat{p}_0 = \sum_{j=0}^{k^\star} \frac{2}{c^2\,\mathrm{vol}(T_{\hat{p}})^{2\varepsilon}} + \sum_{j=0}^{k^\star} \frac{24\eta\cdot\mathrm{vol}(T_{\hat{p}})^{1/2-\varepsilon}}{c}.$$

Using the bound $\mathrm{vol}(T_{\hat{p}_k}) \geq 2^k \deg(s)$ and the geometric sum formula, we have

$$\hat{p}_{k^\star} - \hat{p}_0 \lesssim \sum_{j=0}^{k^\star} \frac{2}{c^2 2^{j2\varepsilon}\deg(s)^{2\varepsilon}} + \sum_{j=0}^{k^\star} \frac{24\eta\cdot\mathrm{vol}(T_{\hat{p}})^{1/2-\varepsilon}}{c} \leq \frac{1}{\varepsilon c^2\deg(s)^{2\varepsilon}} + \sum_{j=0}^{k^\star} \frac{24\eta\cdot\mathrm{vol}(T_{\hat{p}})^{1/2-\varepsilon}}{c}.$$

Using the naive bounds $\mathrm{vol}(T_{\hat{p}}) \leq m$ and $k^\star \lesssim \log n$, we obtain

$$\hat{p}_{k^\star} - \hat{p}_0 \lesssim \frac{1}{\varepsilon c^2\deg(s)^{2\varepsilon}} + \frac{24\eta\cdot m^{1/2-\varepsilon}\log n}{c}.$$

Using similar arguments (sending flow from $t$ to $s$), we see that more than half the volume of the vertices has $\hat{\boldsymbol{p}}$ potential difference at least

$$\frac{1}{\varepsilon c^2\deg(t)^{2\varepsilon}} + \frac{24\eta\cdot m^{1/2-\varepsilon}\log n}{c}$$

to the vertex $t$. Therefore, we can prove the following potential difference upper-bound with respect to $\hat{\boldsymbol{p}}$,

$$\hat{p}(s) - \hat{p}(t) \leq \frac{1}{\varepsilon c^2\deg(s)^{2\varepsilon}} + \frac{1}{\varepsilon c^2\deg(t)^{2\varepsilon}} + \frac{48\eta\cdot m^{1/2-\varepsilon}\log n}{c}.$$

Using the triangle inequality,

$$p(s) - p(t) \leq \hat{p}(s) - \hat{p}(t) + |\hat{p}(s) - p(s)| + |\hat{p}(t) - p(t)| \leq \hat{p}(s) - \hat{p}(t) + 2\eta.$$

We can conclude that

$$\begin{aligned} p(s)-p(t) &\le \frac{1}{\varepsilon c^2 \deg(s)^{2\varepsilon}} + \frac{1}{\varepsilon c^2 \deg(t)^{2\varepsilon}} + \frac{48\eta \cdot m^{1/2-\varepsilon} \log n}{c} + 2\eta \\ &= \frac{1}{\varepsilon c^2 \deg(s)^{2\varepsilon}} + \frac{1}{\varepsilon c^2 \deg(t)^{2\varepsilon}} + \frac{\eta(48 m^{1/2-\varepsilon} \log n + 2c)}{c}. \end{aligned}$$

Noting that $p(s)-p(t) = \mathsf{Reff}(s,t)$ completes the proof. □

We use the above result to complete the proof of our algorithmic result, in which we will choose $\eta$.

**Corollary 14.** *Let $G=(V,E,\boldsymbol{w})$ be a connected weighted graph. If $\deg(v) \ge 1/\alpha$ for all $v \in V$, then for any $0 < \epsilon < 1/2$, there is a cut $(U, U^c)$ such that*

$$\Phi(U) \lesssim \frac{\alpha^{\varepsilon}}{\sqrt{\mathcal{R}_{diam} \cdot \varepsilon}} \cdot \mathrm{vol}(U)^{\varepsilon - 1/2}.$$

*Further, the set $U$ can be found in time $\widetilde{\mathcal{O}}\Big(m \cdot \log\Big(\frac{w(E)}{\min_e w(e)}\Big)\Big)$.*

*Proof.* The proof will be the same as that of Corollary 2. For Corollary 2, we used Theorem 1 to get our non-expanding cut, here we will use Theorem 13. By Lemma 7, we can compute vertices $u, v \in V$ which satisfies,

$$\mathsf{Reff}(u,v) \ge \mathcal{R}_{diam}/3$$

in time $\widetilde{\mathcal{O}}(m)$. As in Corollary 2, we pick

$$c \asymp \sqrt{\frac{\frac{1}{\deg(u)^{2\varepsilon}} + \frac{1}{\deg(v)^{2\varepsilon}}}{\mathcal{R}_{diam} \cdot \varepsilon}}.$$

To contradict the bound from Theorem 13 asymptotically, we need to pick $\eta$ to satisfy

$$\frac{1}{\varepsilon c^2} \cdot \left( \frac{1}{\deg(u)^{2\varepsilon}} + \frac{1}{\deg(v)^{2\varepsilon}} \right) \ge \frac{\eta \cdot (48 \cdot m^{1/2-\varepsilon} \log n + 2c)}{c}.$$

This will allow us to ignore the additional error term we get in the proof of Theorem 13, by settling for a bigger constant that will be hidden in the $\lesssim$-notation.

By the AM-GM inequality, the choice

$$\eta := \frac{1}{\varepsilon \cdot c} \cdot \left( \frac{1}{\deg(u)^{2\varepsilon}} + \frac{1}{\deg(v)^{2\varepsilon}} \right) \cdot \frac{1}{\sqrt{96 \cdot m^{1/2} \log n}}$$

certainly satisfies this. Note that

$$\begin{aligned}
\eta &= \frac{1}{\varepsilon \cdot c} \cdot \left( \frac{1}{\deg(u)^{2\varepsilon}} + \frac{1}{\deg(v)^{2\varepsilon}} \right) \cdot \frac{1}{\sqrt{96 \cdot m^{1/2} \log n}} \\
&\gtrsim \frac{1}{\varepsilon \cdot c^{3/2}} \cdot \frac{1}{w(E)^{2\varepsilon}} \cdot \frac{1}{\sqrt{m^{1/2} \log n}} \\
&\asymp \frac{1}{\varepsilon} \cdot \left( \frac{\mathcal{R}_{diam} \cdot \varepsilon}{\frac{1}{\deg(u)^{2\varepsilon}} + \frac{1}{\deg(v)^{2\varepsilon}}} \right)^{3/4} \cdot \frac{1}{w(E)^{2\varepsilon}} \cdot \frac{1}{\sqrt{m^{1/2} \log n}} \\
&\gtrsim \frac{1}{\varepsilon} \cdot \left( \mathcal{R}_{diam} \cdot \varepsilon \cdot (\min_e w(e))^{2\varepsilon} \right)^{3/4} \cdot \frac{1}{w(E)^{2\varepsilon}} \cdot \frac{1}{\sqrt{m^{1/2} \log n}} \\
&\geq \mathcal{R}_{diam}{}^{\frac{3}{4}} \cdot \frac{(\min_e w(e))^{3\varepsilon/2}}{w(E)^{2\varepsilon}} \cdot \frac{1}{\sqrt{m^{1/2} \log n}}.
\end{aligned}$$

Since the smallest possible $\mathcal{R}_{diam}$ is when all the edges act as parallel resistors between two vertices, we have

$$\mathcal{R}_{diam} \geq \frac{1}{\sum_{e \in E} w(e)} = \frac{1}{w(E)}.$$

Plugging this into the above computation, we get

$$\eta \gtrsim \frac{(\min_e w(e))^{3\varepsilon/2}}{w(E)^{3/4+2\varepsilon}} \cdot \frac{1}{\sqrt{m^{1/2} \log n}}. \tag{A.14}$$

Using this with Lemma 12, we see that we need to pick

$$\zeta \asymp \frac{\eta \cdot (\min_e w(e))^2}{w(E) \cdot \sqrt{m}} \gtrsim \frac{(\min_e w(e))^{2+3\varepsilon/2}}{w(E)^{7/4+2\varepsilon}} \cdot \frac{1}{\sqrt{m^{3/2} \log n}}. \tag{A.15}$$

as the accuracy for the Spielman-Teng solver.

In particular, this means that the Laplacian solver will take time

$$\widetilde{\mathcal{O}}(m \cdot \log(1/\zeta)) = \widetilde{\mathcal{O}}\left( m \cdot \text{polylog}(m) + m \cdot \log\left( \frac{w(E)}{\min_e w(e)} \right) \right) = \widetilde{\mathcal{O}}\left( m \cdot \log\left( \frac{w(E)}{\min_e w(e)} \right) \right).$$

The rest of the proof follows as in Corollary 2. □